Question 1 (5 pts). A bacteria population doubles its size every half an hour. If the initial population is 3 bacteria, write down the differential equation governing the size of the population after $t$ hours, and solve it (Hint: the population grows at a rate proportional to its size).

Solution: First we set up our variables. It seems natural to let $t$ be time in hours and $A(t)$ be the number of bugs present at time $t$. Since the rate of change of the population is proportional to its size, the differential equation for $A$ is

$$A'(t) = kA(t),$$

or

$$A' - kA = 0,$$

where $k$ is a constant that we must find. To solve this equation we multiply the equation by $e^{\int p(t)\,dt} = e^{-kt}$ to obtain

$$\frac{d}{dt} \left( e^{\int p(t)\,dt} A(t) \right) = \frac{d}{dt} \left( e^{-kt} A(t) \right) = 0.$$  

This means that the quantity $e^{-kt} A(t)$ is constant, so

$$A(t) = A_0 e^{kt}.$$ 

$A_0$ in this last equation is the population at time $t = 0$, in this case 3 bugs. Hence $A(t) = 3e^{kt}$.

To find $k$ we use the fact that the population doubles every half an hour. Since at $t = 0$ the population is 3 bugs, at time $t = \frac{1}{2} = .5$ hours the population will be 6 bugs. Hence

$$A(.5) = 3e^{.5k} = 6.$$ 

Solving this last equation for $k$ we find that $k = 2 \ln(2) = \ln(4)$. This means that the solution to the equation we seek is

$$A(t) = 3e^{2\ln(2)t} = 3e^{\ln(4)t} = 3 \times 4^t.$$ 

Question 2 (5 pts). Solve the first order, linear equation

$$ty' + (1 + t)y = t.$$ 

Hint: $\int t e^t \, dt = t e^t - e^t$.

Solution: First we re-write the equation as

$$y' + \left( \frac{1}{t} + 1 \right) y = 1.$$ 

This means that $p(t) = \frac{1}{t} + 1$, so $\int p(t)\,dt = \ln(t) + t$, and $e^{\int p(t)\,dt} = te^t$. Hence

$$\frac{d}{dt} \left( e^{\int p(t)\,dt} y(t) \right) = \frac{d}{dt} \left( te^t y(t) \right) = te^t.$$ 

From here we integrate to obtain

$$te^t y(t) = \int te^t \, dt + \lambda = te^t - e^t + \lambda \text{ so } y(t) = 1 - \frac{1}{t} + \frac{\lambda}{te^t} = 1 - \frac{1}{t} + \frac{\lambda e^{-t}}{t},$$

where $\lambda$ is a constant of integration.