problem 2. The problem says that the function

\[ y(x) = ce^{-2x} + e^{-x} \]

solves the ODE

\[ y' + 2y = e^{-x}, \]

and asks for the value of the constant \( c \) for which \( y(4) = 7 \). To find \( c \) we set

\[ y(4) = ce^{-8} + e^{-4} = 7, \]

from where we find

\[ c = (7 - e^{-4}) e^{8}. \]

Problem 3. We seek two values of \( k \) for which

\[ y(x) = e^{kx} \]

is a solution of

\[ y'' - 8y' + 12y = 0. \]

The way to solve this problem is to plug in the function \( y(x) \) above into the equation. For this we compute

\[ y'(x) = ke^{kx} \quad \text{and} \quad y''(x) = k^2 e^{kx}. \]

From here we deduce

\[ k^2 e^{kx} - 8k e^{kx} + 12e^{kx} = (k^2 - 8k + 12)e^{kx} = 0. \quad (0.1) \]

Since the function \( e^{kx} \) is never zero, the only way the equation (0.1) will hold is if

\[ k^2 - 8k + 12 = 0. \]

This is a quadratic polynomial. Its roots are \( k = 2 \) and \( k = 6 \).

Problem 5. In this problem we know that a sample of radioactive material decays at a rate proportional to the quantity of material present in the sample. We also know that after 940 days the quantity decays by 29%. We must find the half-life of the material, which is defined as the time it takes a sample to decay to half its initial
amount. To do this we first set up our variables. From the information we have we set
\[ t = \text{time in days and } A(t) = \text{amount of material in the sample}. \]

Since the material decays at a rate proportional to the amount of material, the differential equation for \( A(t) \) is
\[ \frac{dA}{dt} = kA, \]
where \( k \) is a constant that we must determine. Before finding \( k \), however, we must solve the equation. To do this we notice that the equation for \( A \) is both first order linear as well as separable. We use either method to integrate. For example we look at the equation as separable, and write
\[ \frac{dA}{A} = k \, dt. \]

Next, we integrate this last equation to obtain
\[ \int \frac{dA}{A} = \int k \, dt + C \]
where \( C \) is a constant of integration. This means that
\[ \ln(A) = kt + C, \]
and solving for \( A \) we find
\[ A(t) = e^{kt+C} = e^C e^{kt}. \]

We rename our constant as \( \lambda = e^C \), to obtain finally
\[ A(t) = \lambda e^{kt}. \]

Note here that
\[ A(0) = \lambda, \]
so the constant \( \lambda \) represents the amount present in the sample at \( t = 0 \), that is, the initial amount.

To find \( k \), we use the information we have. In this case, after 940 days the sample has decayed by 29%, that is, at \( t = 940 \) we have a 71% of the initial amount remaining. Since the initial amount is \( \lambda \), we have
\[ A(940) = .71 \times \lambda. \]

On the other hand, we know that \( A(t) = \lambda e^{kt} \), so we also have
\[ A(940) = \lambda e^{940k}. \]
We then must have
\[ .71 \times \lambda = \lambda e^{940k} \].
We cancel \( \lambda \) in this last equation to obtain
\[ .71 = e^{940k} \].
This last equation is easy to solve for \( k \). The result is
\[ k = \frac{\ln(.71)}{940} \].
The first part of the problem asks about the half-life. Since \( \lambda \) is the initial size of the sample, by definition we seek the value of \( t \) for which
\[ A(t) = \frac{\lambda}{2}, \]
that is we seek the value of \( t \) at which the size of the sample is half the initial size. Since \( A(t) = \lambda e^{kt} \), we set
\[ \lambda e^{kt} = \frac{\lambda}{2} = .5 \lambda, \]
and solve this last equation for \( t \). Note that we already know the value of \( k \). Note also that, again, \( \lambda \) drops out of the equation. Solving for \( t \) this time we obtain
\[ t = \frac{\ln(.5)}{k} = \frac{940 \ln(.5)}{\ln(.71)}, \]
where I used the value of \( k \) we found before.

In part b, the problem asks how ong will it take a sample of 100[\( mg \)] to decay to 83[\( mg \)]. This means that \( \lambda = 100 \) (the initial amount) and we want to find the value of \( t \) for which
\[ A(t) = 83. \]
This gives us the equation
\[ A(t) = 100e^{kt} = 83. \]
Solving for \( t \) we obtain
\[ t = \frac{\ln(.83)}{k} = \frac{940 \ln(.83)}{\ln(.71)}. \]

Problem 6. In this problem we are given the ODE
\[ \frac{dP}{dt} = \frac{6}{700} P(7 - P), \]
along with the initial condition $P(0) = 2$.

The first two questions ask us to determine the range of $P$ in which $P(t)$ is increasing and decreasing respectively. To do this we look at the expression on right-hand side of the equation and determine when is this expression positive and negative. The expression we have is

$$f(P) = P(7 - P).$$

It is easy to notice that

$$f(P) > 0 \text{ for } P \in (0, 7) \text{ and } f(P) < 0 \text{ for } P > 7.$$  

We conclude that $P(t)$ is increasing when $0 < P < 7$ and $P(t)$ is decreasing when $P > 7$.

For problem 6, part C, we need to solve the differential equation

$$\frac{dP}{dt} = \frac{6}{700}P(7 - P).$$

We notice that this is a separable, first order equation. To solve it, we write

$$\frac{dP}{P(7 - P)} = \frac{6}{700}dt,$$

and then integrate. To integrate the left-hand side we need to use partial fractions, or just look at the problem and notice that

$$\frac{1}{P(7 - P)} = \frac{1}{7} \left( \frac{1}{P} + \frac{1}{7 - P} \right).$$

This means that we need to integrate

$$\frac{1}{7} \left( \frac{1}{P} + \frac{1}{7 - P} \right) dP = \frac{6}{700} dt.$$

Integrating, we obtain

$$\ln \left( \frac{P}{7 - P} \right) = \frac{6}{100} t + C = \frac{3}{50} t + C.$$  

Exponentiating this equation we obtain

$$\frac{P}{7 - P} = e^{\frac{3}{50} t} \times e^C.$$  

Again, we rename our constant as $\lambda = e^C$, so we get

$$\frac{P}{7 - P} = \lambda e^{\frac{3}{50} t}.$$
To determine the value of $\lambda$ we recall that $P(0) = 2$, so

$$\frac{2}{7 - 2} = \frac{2}{5} = \lambda.$$  

We have so far that

$$\frac{P}{7 - P} = \frac{2}{5} e^{\frac{3}{5}t}.$$  

Finally, we solve this last equation for $P$. We do this as follows:

$$-1 + \frac{7}{7 - P} = \frac{P}{7 - P} = \frac{2}{5} e^{\frac{3}{5}t},$$

so that

$$\frac{7}{7 - P} = \frac{P}{7 - P} = 1 + \frac{2}{5} \frac{3}{5} e^{\frac{3}{5}t} = \frac{5 + 2 e^{\frac{3}{5}t}}{5}.$$  

From here we obtain

$$\frac{35}{5 + 2 e^{\frac{3}{5}t}} = 7 - P,$$

so

$$P(t) = 7 - \frac{35}{5 + 2 e^{\frac{3}{5}t}}.$$  

This gives us $P(t)$ for any $t$. For example

$$P(2) = 7 - \frac{35}{5 + 2 e^{\frac{3}{5}2}} = 7 - \frac{35}{5 + 2 e^{\frac{6}{5}}}.$$  

**Problem 7.** In this problem we consider the ODE

$$\frac{dP}{dt} = c \ln \left( \frac{K}{P} \right) P.$$  

This equation models the growth of a limited population and is called Gompertz equation. Assuming, for example, $c = .05$, $K = 5000$, and that the initial population is $P(0) = 600$, we are asked first to solve the ODE. To do this we start by writing

$$\frac{dP}{P \ln \left( \frac{K}{P} \right)} = c \, dt,$$

which is the same as

$$\frac{dP}{P (\ln(K) - \ln(P))} = c \, dt,$$

and then integrate. This gives you the following result:

$$- \ln \left( \ln(K) - \ln(P) \right) = c t + L.$$
where \( L \) is a constant of integration. At this point we need two things: first, we need to solve for \( P \) as a function of \( t \). Second, we need to determine the value of the constant \( L \). Let us solve for \( P(t) \) first.

We exponentiate the equation

\[ -\ln(\ln(K) - \ln(P)) = ct + L \]

to obtain

\[ \frac{1}{\ln(K) - \ln(P)} = e^{ct}e^{L}. \]

As we often do, I will rename the constants according to \( \lambda = e^{-L} \). Hence we have so far

\[ \ln(K) - \ln(P) = \lambda e^{-ct}. \]

From here we can now solve for \( P \):

\[ \ln(P) = \ln(K) - \lambda e^{-ct}, \]

so

\[ P(t) = \frac{K}{e^{\lambda e^{-ct}}}. \]

To determine the value of \( \lambda \) we can use this last equation, but it is actually easier to recall that

\[ \ln(P) = \ln(K) - \lambda e^{-ct}. \]

Since we know that \( c = .05, K = 5000, \) and \( P(0) = 600, \) the value of \( \lambda \) comes directly from here:

\[ \lambda = \ln(600) - \ln(5000) = \ln\left(\frac{3}{25}\right). \]

This determines \( P(t) \) completely. For part b we need to find the limit

\[ \lim_{t \to \infty} P(t). \]

For this we go back to the solution

\[ P(t) = \frac{K}{e^{\lambda e^{-ct}}}. \]

Note that, as \( t \to \infty \), we have \( e^{-ct} \to 0 \), so that \( e^{\lambda e^{-ct}} \to 1 \). We conclude that

\[ \lim_{t \to \infty} P(t) = K = 5000. \]

Finally we need to determine the value of \( P \) at which the function \( P(t) \) is growing fastest. For this we notice that

\[ \frac{dP}{dt} = c \ln\left(\frac{K}{P}\right)P. \]
means that the value of $\frac{dP}{dt}$ is given by the right-hand side of this equation. Hence $P$ will grow fastest when the right-hand side of this last equation is greatest. Since $c = .05 > 0$, what we need then is to find the maximum value of the function

$$\psi(P) = \ln\left(\frac{K}{P}\right) P = P(\ln(K) - \ln(P)).$$

This is of course standard. We compute $\psi'(P)$ and set it equal to zero. We obtain

$$\psi'(P) = \ln\left(\frac{K}{P}\right) - 1 = 0,$$

which means $P(t)$ is growing fastest when $P = \frac{K}{e}$. 

**Problem 8.** In this problem we have an object of mass $m = 5 \text{ [kg]}$ is thrown up with an initial speed of $v_0 = 90 \text{ [m/s]}$. The gravity of the place is $g = 9.8 \text{ [m/s}^2\text{]}, and there is an air resistance given by the constant $k$. To find a formula for the velocity $v$ as a function of $t$, we recall Newton’s second Law, that says that mass times acceleration equals the sum of all forces acting on an object. Now, the acceleration of an object is the derivative of its velocity, so our equation becomes

$$mv' = -mg - kv.$$

This is a first order linear equation, but it is also separable. We can choose either method to solve it. I will think of it as a separable equation. So, I write the equation as

$$\frac{dv}{dt} = -\left(\frac{k}{m}v + g\right) = -\frac{k}{m}\left(v + \frac{mg}{k}\right),$$

from where I get

$$\frac{dv}{(v + \frac{mg}{k})} = -\frac{k}{m} dt.$$

This is easy to integrate:

$$\ln\left(v + \frac{mg}{k}\right) = -\frac{k}{m}t + C,$$

where $C$ is a constant of integration. Solving for $v$ from here is a piece of cake:

$$v(t) = -\frac{mg}{k} + e^{-\frac{k}{m}t}e^C.$$

As always, we rename our constant as $\lambda = e^C$. This means we obtain

$$v(t) = -\frac{mg}{k} + \lambda e^{-\frac{k}{m}t}.$$
Since we know that $v(0) = v_0 = 90$, we obtain

$$-\frac{mg}{k} + \lambda = v_0,$$

which means $\lambda = v_0 + \frac{mg}{k}$. We finally arrive at

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{kt}{m}} = v_0e^{-\frac{kt}{m}} - \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right),$$

and we know all the constants that appear in the last expression for $v(t)$, so this determines $v(t)$ completely.

The second part of this question asks what happens to $v(t)$ as $k \to 0$. In other words, we need to compute

$$\lim_{k \to 0} v(t).$$

Since $v(t)$ is the sum of two terms, we can find the limits of these separately and then add the results. First, it is clear that

$$\lim_{k \to 0} v_0e^{-\frac{kt}{m}} = v_0.$$

Next, we need to find

$$\lim_{k \to 0} \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right) = g \lim_{k \to 0} \frac{m}{k}\left(1 - e^{-\frac{kt}{m}}\right).$$

Setting $x = \frac{k}{m}$, we notice that as $k \to 0$ we have $x \to 0$. Hence we need to find the limit

$$\lim_{x \to 0} \frac{1 - e^{-xt}}{x}.$$

Here we notice that this limit is of the form $\frac{0}{0}$ so we use l’Hôpital’s rule to obtain

$$\lim_{x \to 0} \frac{1 - e^{-xt}}{x} = \frac{t}{1} = t.$$

Putting this all together we arrive at the conclusion that

$$\lim_{k \to 0} v(t) = v_0 - gt.$$

**Problem 9.** First we declare our variables:

$T =$ temperature of the object. $T_m =$ temperature of the room. $t =$ time.

Now we know that the temperature of the object changes at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. Letting $k$ be the proportionality constant we obtain

$$\frac{dT}{dt} = k(T - T_m).$$
This is both a first order linear equation and a separable equation. In my version of this problem I have that $T_m = 78$, so to solve this equation, for example, we write

$$\frac{dT}{T - 78} = kdt,$$

and then integrate. This gives us

$$\ln(T - 78) = tk + C,$$

so

$$T = 78 + e^{kt+C} = 78 + C e^{kt}.$$

It is easier, as we have done several times in class, to set

$$\lambda = e^C,$$

so the solution to our ODE now looks as follows:

$$T(t) = 78 + \lambda e^{kt}.$$

We need now to find the values of the constants $\lambda$ and $k$. To do this we use information given in the problem. My version of the problem says that $T(0) = 195$, and also $T(3) = 180$. The first of these data tells us that

$$T(0) = 78 + \lambda = 195,$$

so $\lambda = 117$. On the other hand, the second datum says

$$T(3) = 180 = 78 + 117 e^{3k},$$

so we can solve for $k$ to obtain

$$\frac{\ln\left(\frac{102}{117}\right)}{3} = k.$$

Hence we know the function $T(t)$ completely. Finally my version of the problem asks me to find the value of $t$ for which $T(t) = 150$. To find $t$ I set up the equation

$$78 + 117 e^{kt} = 150$$

and solve for $t$. This last equation yields

$$t = \frac{\ln\left(\frac{72}{117}\right)}{k}.$$

Since we actually know the value of $k$, this solves the problem.
Problem 10. This problem says that a bacteria culture doubles its population every 20\,[\text{min}], and that the initial population is 4 bugs. Then it asks to find the population of the culture after \( t \) hours, the population after exactly 8 hours, and determine the time it takes the culture to have 32 bugs. To answer these questions, as always, we first we set up our variables. It seems natural to let \( t \) be time in hours and \( A(t) \) be the number of bugs present at time \( t \). Since the rate of change of the population is proportional to its size, the differential equation for \( A \) is \( A'(t) = kA(t) \), or \( A' - kA = 0 \), where \( k \) is a constant that we must find. To solve this equation we multiply the equation by 

\[ e^\int p(t)\,dt = e^{-kt} \]

to obtain

\[ \frac{d}{dt}\left(e^\int p(t)\,dt\,A(t)\right) = \frac{d}{dt}(e^{-kt}A(t)) = 0. \]

This means that the quantity \( e^{-kt}A(t) \) is constant, so

\[ A(t) = A_0 e^{kt}. \]

\( A_0 \) in this last equation is the population at time \( t = 0 \), in this case 4 bugs. Hence \( A(t) = 4e^{kt} \).

To find \( k \) we use the fact that the population doubles every 20\,[\text{min}]. Since at \( t = 0 \) the population is 4 bugs, 20\,[\text{min}] later, that is, at time \( t = \frac{1}{3} \) hours the population will be 8 bugs. Hence

\[ A\left(\frac{1}{3}\right) = 4e^{\frac{k}{3}} = 8. \]

Solving this last equation for \( k \) we find that \( k = 3 \ln(2) = \ln(8) \). This means that the solution to the equation we seek is

\[ A(t) = 4e^{3\ln(2)t} = 4e^{\ln(8)t} = 4 \times 8^t. \]

From here we obtain

\[ A(8) = 4 \times 8^8 = 2^{26} \text{ bugs}. \]

Finally we want to know the value of \( t \) for which \( A(t) = 32 \). We can just reason this out as follows: if initially we have 4 bugs, and the population doubles every 20\,[\text{mins}], after 20 minutes we will have 8 bugs, after 40 we will have 16 and after an hour we will have 32 bugs. Since we measure time in hours, the answer is \( t = 1 \). We could also set up the equation

\[ 32 = A(t) = 4 \times 8^t \]

or

\[ 32 = 4 \times 8^t, \]
and solve for $t$. The solution to this last equation is clearly $t = 1$, which agrees with our previous reasoning.

**Problem 11.** This problem asks to solve the differential equation

$$11x - 4y\sqrt{x^2 + 1}\frac{dy}{dx} = 0,$$

subject to the initial condition $y(0) = 7$.

**Solution:** First we notice this is a separable equation. We re-write it as

$$y\frac{dy}{dx} = \frac{11}{4} \frac{x}{\sqrt{x^2 + 1}}.$$

Then we integrate

$$\int y\,dy = \frac{11}{4} \int \frac{x}{\sqrt{x^2 + 1}}\,dx + C.$$

For the integral on the left it is convenient to use the substitution $u = x^2 + 1$, and $du = 2x\,dx$. Hence

$$\int \frac{x}{\sqrt{x^2 + 1}}\,dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2 + 1}}\,dx = \frac{1}{2} \int \frac{1}{\sqrt{u}}\,du = \sqrt{u} = \sqrt{x^2 + 1}.$$

This means that

$$\frac{y^2}{2} = \frac{11}{4} \sqrt{x^2 + 1} + C,$$

where $C$ is a constant of integration. To find $C$ we use the fact that $y(0) = 7$ to obtain

$$\frac{49}{2} = \frac{11}{4} + C \quad \text{so} \quad C = \frac{87}{4}.$$

Our solution is then

$$y = \sqrt{2 \left( \frac{11}{4} \sqrt{x^2 + 1} + \frac{87}{4} \right)}.$$

**Problem 12.** Here we want to solve

$$4\frac{dy}{dt} + y = 24t,$$

with $y(0) = 8$.

**Solution:** This equation, and those in the remaining two problems, is a first order linear equation. The general method to solve these equations goes like this: first we consider the equation

$$y' + p(t)y = f(t),$$

...
and multiply the whole thing by \( e^{\int p(t) \, dt} \).

This makes the left hand side into a perfect derivative:

\[
\frac{d}{dt} \left( e^{\int p(t) \, dt} y \right) = e^{\int p(t) \, dt} \, y' + p(t) e^{\int p(t) \, dt} \, y = e^{\int p(t) \, dt} \, f(t),
\]

so we obtain

\[
\frac{d}{dt} \left( e^{\int p(t) \, dt} y \right) = e^{\int p(t) \, dt} \, f(t).
\]

To solve this equation for \( y \) we must integrate. In **Problem 12** we have

\[
4 \frac{dy}{dt} + y = 24t,
\]

which we re-write as

\[
\frac{dy}{dt} + \frac{y}{4} = 6t.
\]

This means that

\[
p(t) = \frac{1}{4}, \quad \text{so} \quad \int p(t) \, dt = \frac{t}{4} \quad \text{and then} \quad e^{\int p(t) \, dt} = e^{\frac{t}{4}}.
\]

Note that in this integral **we do not need a constant of integration**. Once we have this we write

\[
\frac{d}{dt} \left( e^{\frac{t}{4}} y \right) = 6te^{\frac{t}{4}},
\]

and integrate to obtain

\[
e^{\frac{t}{4}} y = 6 \int te^{\frac{t}{4}} \, dt + C.
\]

Note that **we do need a constant of integration here**. To compute the integral

\[
\int te^{\frac{t}{4}} \, dt
\]

we go by parts. Let \( u = t \), so \( du = dt \), and \( dv = e^{\frac{t}{4}} \), so \( v = 4e^{\frac{t}{4}} \). Then

\[
\int te^{\frac{t}{4}} \, dt = uv - \int v \, du = 4te^{\frac{t}{4}} - 4 \int e^{\frac{t}{4}} \, dt = 4te^{\frac{t}{4}} - 16e^{\frac{t}{4}}.
\]

So far, then, we have

\[
e^{\frac{t}{4}} y = 6 \left( 4te^{\frac{t}{4}} - 16e^{\frac{t}{4}} \right) + C = 32e^{\frac{t}{4}} (t - 4) + C.
\]

From here we obtain

\[
y(t) = 24 (t - 4) + C e^{-\frac{t}{4}}.
\]
To determine the constant $t$ we use the condition $y(0) = 8$, which gives us

\[ 8 = 24(-4) + C, \]

so $C = 104$. The solution to our problem is

\[ y(t) = 24(t - 4) + 104e^{-\frac{t}{4}}. \]

**Problem 13.** Here we consider the equation

\[ \frac{dy}{dt} - 2ty = 3t^2e^{t^2}, \]

and the condition $y(0) = -1$.

**Solution:** The equation is already in the form $y' + p(t)y = f(t)$. Here

\[ p(t) = -2t \quad \text{so} \quad e^\int p(t)\, dt = e^{-t^2}. \]

This allows us to write

\[ \frac{d}{dt} \left( e^{-t^2} y \right) = 3t^2. \]

Integrating this we obtain

\[ e^{-t^2} y = t^3 + C, \]

where $C$ is a constant of integration. We have so far

\[ y(t) = (t^3 + C)e^{t^2}. \]

We use finally the condition $y(0) = -1$ to obtain

\[ y(0) = C = -1 \]

so $C = -1$. The solution to the problem is

\[ y(t) = (t^3 - 1)e^{t^2}. \]

**Problem 14.** Our last problem is to solve the ODE

\[ 9(t + 1) \frac{dy}{dt} - 8y = 8t, \]

with the initial condition $y(0) = 10$. 
Solution: Again, we first write the equation in the form $y' + p(t)y = f(t)$. This means
\[
\frac{dy}{dt} - \frac{8}{9(t+1)}y = \frac{8t}{9(t+1)}.
\]
Hence
\[
p(t) = -\frac{8}{9(t+1)} \quad \text{so} \quad e^{\int p(t)dt} = \frac{1}{(1+t)^{\frac{8}{9}}}.
\]
The equation then becomes
\[
\frac{d}{dt} \left( \frac{y}{(1+t)^{\frac{8}{9}}} \right) = \frac{8t}{9(1+t)^{\frac{17}{9}}}.
\]
Here we need to find the integral
\[
\int \frac{8t}{9(1+t)^{\frac{17}{9}}} \, dt = \frac{8}{9} \int \frac{t}{(1+t)^{\frac{17}{9}}} \, dt.
\]
For the last integral we could go by parts, but we can also notice that
\[
\int \frac{t}{(1+t)^{\frac{17}{9}}} \, dt = \int \frac{1+t-1}{(1+t)^{\frac{17}{9}}} \, dt
\]
\[
= \int \frac{1}{(1+t)^{\frac{17}{9}}} \, dt - \int \frac{1}{(1+t)^{\frac{17}{9}}} \, dt
\]
\[
= 9(1+t)^{\frac{1}{9}} + \frac{9}{8(1+t)^{\frac{8}{9}}}.
\]
From this we obtain
\[
\frac{y}{(1+t)^{\frac{8}{9}}} = \frac{8}{9} \left( 9(1+t)^{\frac{1}{9}} + \frac{9}{8(1+t)^{\frac{8}{9}}} \right) + C,
\]
so
\[
y(t) = 8(1+t) + 1 + C(1+t)^{\frac{8}{9}}.
\]
We finally use the condition $y(0) = 10$ to obtain
\[
10 = 8 + 1 + C,
\]
so $C = 1$. The solution we seek is
\[
y(t) = 8(1+t) + 1 + (1+t)^{\frac{8}{9}}.
\]