CONJUGACY CLASSES OF SMALL SIZES
IN THE LINEAR AND UNITARY GROUPS

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Abstract. Using the classical results of Wall on the parametrization and sizes of (conjugacy) classes of finite classical groups, we present some gap results for the class sizes of the general linear groups and general unitary groups as well as their variations. In particular, we identify the classes in \(GL_n(q)\) of size up to \(q^{4n-8}\) and classes in \(GU_n(q)\) of size up to \(q^{4n-9}\). We then apply these gap results to obtain some bounds and limits concerning the zeta-type function encoding the conjugacy class sizes of these groups.

1. Introduction and results

The problem of determining the irreducible characters (including both complex and Brauer characters) of relatively small degrees of finite classical groups and more generally finite groups of Lie type has been studied extensively over the last two decades. The results obtained have proved very useful in various applications. The irreducible complex characters of finite Lie-type groups of small degrees were studied in [12, 16] and more recently in [15]. In these papers, the irreducible complex characters of degrees up to a certain bound are classified and it is proved that there is a large gap between the degrees of these characters and the next degree. For that reason, these results are often referred to as the gap results on character degrees of finite groups.

Here we turn our attention to the problem of determining the conjugacy classes of relatively small sizes of finite classical groups.

Conjugacy classes of finite classical groups have been studied since the classical work of Wall [18]. There indeed has been several ways to formulate and estimate the class sizes of finite classical groups. For instance, besides the well-known class size formulas of Wall, it is shown by Liebeck and Shalev [10] that the size of the class of an element \(g\) in a classical group \(G\) can be presented in terms of the codimension of the largest eigenspace of \(g\) on the natural \(G\)-module. Burness in the series
of papers [1, 2, 3, 4] has used this formulation to estimate the class sizes of prime order elements and then to study the fixed point ratios in actions of finite classical groups. Recently, Fulman and Guralnick [8] have provided an upper bound for the largest class size in various finite classical groups.

We approach the problem of small size classes as follows. Let \( G \) be a finite classical group and let \( g \in G \). Then \( g \) has a unique Jordan decomposition \( g = su = us \) where \( s \in G \) is semisimple and \( u \in G \) is unipotent. That is, \( s \) is diagonalizable over the algebraic closure of the underlying field of \( G \) and all eigenvalues of \( u \) are equal to 1.

Let \( \text{cl}_G(g) \) denote the conjugacy class of \( G \) containing \( g \). The Jordan decomposition of conjugacy classes says that \( \text{cl}_G(g) \leftrightarrow (\text{cl}_G(s), \text{cl}_{C_G}(s)(u)) \) is a one-to-one correspondence, where \( C_G(s) \) is the centralizer of \( s \) in \( G \). Furthermore,

\[
|\text{cl}_G(g)| = |\text{cl}_G(s)||\text{cl}_{C_G}(s)(u)|.
\]

The structure of the centralizers of semisimple elements in finite classical groups are well-known and due to Carter [6]. These centralizers are isomorphic (or close) to a product of classical groups of lower dimension. Therefore, in order to find small size classes, one should identify the semisimple classes and unipotent classes of small size. The formulas describing the class sizes of finite classical groups are often complicated (see (2.1) and (2.2) below) and it is difficult to derive the classes of size up to a certain bound. However, when reducing to unipotent and semisimple classes, we are able to estimate those formulas.

Our first main results are:

**Theorem 1.1.** Let \( n \geq 6 \) and \( q \) be a prime power. Then \( \text{GL}_n(q) \) has

(i) \( q - 1 \) (central) classes of size 1,
(ii) \( q - 1 \) classes of size \( (q^{n-1} - 1)(q^n - 1)/(q - 1) \), and
(iii) \( q^2 - 3q + 2 \) (semisimple) classes of size \( q^{n-1}(q^n - 1)/(q - 1) \).

Moreover, all other classes have sizes greater than \( q^{4n-8} \).

**Theorem 1.2.** Let \( n \geq 5 \) and \( q \) be a prime power. Then \( \text{GU}_n(q) \) has

(i) \( q + 1 \) (central) classes of size 1,
(ii) \( q + 1 \) classes of size \( (q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)/(q + 1) \), and
(iii) \( q^2 + q \) (semisimple) classes of size \( q^{n-1}(q^n - (-1)^n)/(q + 1) \).

Moreover, all other classes have sizes greater than \( q^{4n-9} \).

Theorem 1.1 and Theorem 1.2 will be proved in Sections 3 and 5, respectively. We also derive in Sections 4 and 6 some corresponding
gap results for other variations of the linear and unitary groups. We note that the condition on $n$ is necessary and it is indeed not difficult to work out all the class sizes of the classical groups in low dimension. The symplectic and orthogonal groups require more delicate treatment and as the paper is long enough, we leave these groups to another time.

For a finite group $G$, let $\mathcal{C}(G)$ denote the set of all conjugacy classes of $G$. Following [11], we define the zeta-type function encoding the conjugacy class sizes of $G$ as follows:

$$\zeta^G(t) = \sum_{C \in \mathcal{C}(G)} |C|^{-t} \text{ for } t \in \mathbb{R}.$$  

This function, together with its dual analogue for character degrees, have appeared in several contexts. For instance, it was used by Liebeck and Shalev in [11] §5 and §7 to obtain some probabilistic results on base sizes for primitive actions of almost simple classical groups of dimensions high enough. In [5, 10], a slightly different zeta-type function where the sum is taken over all conjugacy classes of elements of prime order in $G$ was defined and applied to obtain upper bounds on base sizes for almost simple primitive permutation actions. For more details on this, we refer the reader to [5, 10, 11] and references therein.

Let $k_s(G)$ denote the number of classes of $G$ of size $s$. The zeta-type function can be rewritten as

$$\zeta^G(t) = \sum_{s \in \mathbb{N}} k_s(G)s^{-t}.$$  

When $t$ is positive, we see that the main contribution of the above sum is the terms where the size $s$ is small. Therefore, knowing the conjugacy classes of small sizes of $G$ will provide a good estimation for $\zeta^G(t)$.

Using the obtained results on conjugacy classes of relatively small sizes of the linear and unitary groups, we prove in Section 7 some results on upper bounds and limits involving their associated zeta-type functions. One such result is the following:

**Theorem 1.3.** Let $G = \text{GL}_n(q)$ or $\text{GU}_n(q)$ where $n \geq 6$. If $t > 1/3$, then

$$\frac{\zeta^G(t)}{q} \to 1 \text{ as } q \to \infty.$$  

Moreover, the convergence is uniformly in $n$. 
2. Preliminaries

2.1. Linear groups. Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}_q$ of $q$ elements. Let $G$ be the group of invertible linear transformations of $V$, i.e. $G$ is the general linear group $\text{GL}_n(q)$. Let $\mathbb{F}_q(x)$ be the ring of polynomials over $\mathbb{F}_q$ with variable $x$. For each element $g$ in $G$, define an action of $\mathbb{F}_q(x)$ on $V$ by $v \cdot x = vg$ for all $v \in V$. This makes $V$ become an $\mathbb{F}_q(x)$-module. Denote this module by $V_g$. It is easy to see that two elements $g$ and $h$ of $G$ are conjugate if and only if $V_g$ and $V_h$ are isomorphic as $\mathbb{F}_q(x)$-modules, see [13].

Let $\mathcal{M}$ be the set of monic irreducible polynomials in $\mathbb{F}_q[x]$ different from $x$ and let $\mathcal{P}$ be the set of partitions. Since $\mathbb{F}_q(x)$ is a principle domain, the $\mathbb{F}_q(x)$-module $V_g$ is a direct sum of modules of the form $\mathbb{F}_q(x)/(f)^m$ where $m \geq 1$ and $(f)$ is the ideal generated by $f \in \mathcal{M}$. In other words, $V_g \cong \bigoplus_{f \in \mathcal{M}, i \in \mathbb{N}} \mathbb{F}_q(x)/(f)^{\lambda(f)}$, where $\lambda(f) = (\lambda_1(f), \lambda_2(f), \ldots)$ is a partition. Therefore, we get a function $\lambda : \mathcal{M} \to \mathcal{P}$ such that $\sum_{f \in \mathcal{M}} \deg(f)|\lambda(f)| = n$. Denoting this function $\lambda$ by $\lambda_g$, we now obtain a one-to-one correspondence $\mathcal{C}(g) \leftrightarrow \lambda_g$ between the set of conjugacy classes of $G$ and the set of partition-valued functions $\lambda$ such that

$$\sum_{f \in \mathcal{M}} \deg(f)|\lambda(f)| = n.$$  

For reader convenience, we recall some standard notations and terminologies of partitions. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a partition, i.e. $\alpha_1 \geq \alpha_2 \geq \cdots$. Each positive number $\alpha_i$ is called a part of $\alpha$. The size of $\alpha$, which is $\alpha_1 + \alpha_2 + \cdots$ will be denoted by $|\alpha|$. Let $m_i(\alpha)$ be the number of parts of $\alpha$ of size $i$ and let $\alpha'$ be the dual partition of $\alpha$ in the sense that $\alpha'_i = m_i(\alpha) + m_{i+1}(\alpha) + \cdots$. If a part $\alpha$ in a partition is repeated $k$ times, we will write $(\ldots, \alpha^k, \ldots)$ instead of $(\ldots, \alpha, \ldots, \alpha, \ldots)$.

It is well known that each conjugacy class of $G$ is determined by its rational canonical form. The one-to-one correspondence between rational canonical forms and partition-valued functions on $\mathcal{M}$ is known, see [7] for instance. From this correspondence, we see that an element $g \in G$ is semisimple if and only if all parts of partitions $\lambda_g(f), f \in \mathcal{M}$, are at most 1. Also, $g \in G$ is unipotent if and only if $|\lambda_g(f)| = 0$ for all $f \in \mathcal{M}$ except when $f$ is the polynomial $x - 1$. In particular, unipotent classes of $G$ are parameterized by partitions of $n$.

Suppose that $a$ is a nonzero real number, $m$ is a positive integer, and $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a partition. Let $(a)_m$ denote $(1-a)(1-a^2) \cdots (1-a^m)$ and $h(a, \alpha)$ denote $a^{\Sigma_1(\alpha'_1)} \prod_i (1/a)^{m_i(\alpha)}$. We also set $(a)_0 = 1$. If $\lambda = \lambda_g$ is the partition-valued function corresponding to $g \in G$, the size of the
centralizer \( C_G(g) \) is (see [7, 9, 14]):

\[
(2.1) \prod_{f \in \mathcal{M}} q^{\deg(f) \sum_i (\lambda(f))^2} \prod_i (1/q^{\deg(f)})^{m_i(\lambda(f))} = \prod_{f \in \mathcal{M}} h(q^{\deg(f)}, \lambda(f)).
\]

A lower bound for \( h(a, \alpha) \) where \( \alpha \in \mathcal{P} \) is obtained by Fulman and Guralnick in [8, Lemma 6.1]. Since we are working on conjugacy classes of small sizes, we need to maximize \( h(a, \alpha) \).

**Lemma 2.1.** Let \( \alpha \) be a partition of size \( k \geq 4 \). If \( a \geq 2 \) and \( \alpha \) is not \((1^k)\) or \((2, 1^{k-2})\), then \( h(a, \alpha) \leq h(a, (2^2, 1^{k-4})) \).

**Proof.** Suppose that \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \neq (1^k) \) where \( \alpha_1 \geq \alpha_2 \geq \cdots \alpha_m \geq 1 \) and \( i \) is the largest index so that \( t := \alpha_i \geq 2 \). Consider the partition \( \beta = (\beta_1, \beta_2, ..., \beta_m) \) of size \( k \) such that \( \beta_j = \alpha_j \) for all \( j \neq i \) and \( \beta_i = \alpha_i - 1 \). We will show as follows that \( h(a, \beta) > h(a, \alpha) \). If \( t = 2 \), then

\[
\frac{h(a, \beta)}{h(a, \alpha)} = \frac{a^{\Sigma_i (\beta_i')^2} (1/a)_{m_1(\alpha) + 2} (1/a)_{m_2(\alpha) - 1}}{a^{\Sigma_i (\alpha_i')^2} (1/a)_{m_1(\alpha)} (1/a)_{m_2(\alpha)}}
\geq \frac{a^2 (1 - 1/a^{m_1(\alpha) + 1})(1 - 1/a^{m_1(\alpha) + 2})}{1 - 1/a^{m_2(\alpha)}}
> a^2 (1 - 1/a)^2 \geq 1.
\]

Similarly, if \( t > 2 \) then

\[
\frac{h(a, \beta)}{h(a, \alpha)} = \frac{a^{\Sigma_i (\beta_i')^2} (1/a)_{m_1(\alpha) + 1} (1/a)_{m_2(\alpha) - 1} (1/a)_{m_{t-1}(\alpha) + 1}}{a^{\Sigma_i (\alpha_i')^2} (1/a)_{m_1(\alpha)} (1/a)_{m_2(\alpha)} (1/a)_{m_{t-1}(\alpha)}}
\geq \frac{a^2 (1 - 1/a^{m_1(\alpha) + 1})(1 - 1/a^{m_{t-1}(\alpha) + 1})}{1 - 1/a^{m_{t}(\alpha)}}
> a^2 (1 - 1/a)^2 \geq 1.
\]

Since \( \alpha \) is not \((1^k)\) or \((2, 1^{k-2})\), after some transforms similar to the one above, we will get to either the partition \((2^2, 1^{k-4})\) or \((3, 1^{k-3})\). Therefore,

\[
h(a, \alpha) \leq \max\{h(a, (2^2, 1^{k-4})), h(a, (3, 1^{k-3}))\}.
\]

Observe that

\[
h(a, (2^2, 1^{k-4})) = a^{k^2 - 4k + 8}(1/a)_{k-4}(1 - 1/a)(1 - 1/a^2)
= \frac{a^2 (1 - 1/a^2)}{1 - 1/a^{k-3}} h(a, (3, 1^{k-3}))
> a^2 (1 - 1/a^2) h(a, (3, 1^{k-3}))
> h(a, (3, 1^{k-3}))
\]
2.2. Unitary groups. The general unitary group GU$_n(q)$ is the subgroup of all transformations of GL$_n(q^2)$ that fix a non-singular Hermitian form on $\mathbb{F}_{q^2}$. The following description of conjugacy classes of GU$_n(q)$ and their sizes is due to Wall [18]. Let $\mathcal{N}$ be the set of monic irreducible polynomials in $\mathbb{F}_{q^2}[x]$ different from $x$. For each $f(x) = x^{\deg(f)} + a_{\deg(f)-1}x^{\deg(f)-1} + \cdots + a_1x + a_0$ in $\mathcal{N}$, the conjugate polynomial of $f$, denoted by $\tilde{f}$, is defined to be

$$\tilde{f}(x) = x^{\deg(f)} + (a_1/a_0)^qx^{\deg(f)-1} + \cdots + (a_{\deg(f)-1}/a_0)^qx + (1/a_0)^q.$$ 

Similar to the general linear groups, there is a one-to-one correspondence $\text{cl}(g) \mapsto \lambda_g$ between the set of conjugacy classes of GU$_n(q)$ and the set of partition-valued functions $\lambda : \mathcal{N} \rightarrow \mathcal{P}$ such that $\sum_{f \in \mathcal{N}} \deg(f)|\lambda(f)| = n$ and $\lambda(f) = \lambda(\tilde{f})$. The order of the corresponding centralizer $C_{GU_n(q)}(g)$ is (see [8, 18]):

$$\prod_{f \in \mathcal{N}, f = \tilde{f}} q^{\deg(f)} \sum_{l} \lambda(f)^l \prod_{i} (-1/q^{\deg(f)})^{m_i(\lambda(f))} \prod_{i} (1/q^{\deg(f)})^{m_i(\lambda(f))} \frac{1}{q^{\deg(f)}}.$$

where $q \mapsto q^2$ means that all occurrences of $q$ are replaced by $q^2$. If $l(a, \alpha)$ denotes $a^{\sum_i(\alpha_i)^2} \prod_i (-1/a)m_i(\alpha)$, then the above centralizer order is

$$\prod_{f \in \mathcal{N}, f = \tilde{f}} l(q^{\deg(f)}, \lambda(f)) \prod_{f \in \mathcal{N}, f \neq \tilde{f}} h(q^{\deg(f)}, \lambda(f))$$

A lower bound for $l(a, \alpha)$ where $\alpha \in \mathcal{P}$ is given in [8, Lemma 6.5]. Here again we need to maximize $l(a, \alpha)$.

**Lemma 2.2.** Let $\alpha$ be a partition of size $k \geq 4$. If $a \geq 2$ and $\alpha$ is not $(1^k)$ or $(2, 1^{k-2})$, then $l(a, \alpha) \leq l(a, (2^2, 1^{k-4}))$.

**Proof.** This is similar to the proof of Lemma 2.1.

We also need the following known result on the centralizers of semisimple elements in the unitary groups.

**Lemma 2.3** (6 [17]). Let $s$ be a semisimple element of GU$_n(q)$ with characteristic polynomial given by the product

$$\prod_{i=1}^{t} (f_i \tilde{f}_i)^{a_i} \prod_{i=1}^{r} g_i^{b_i}$$

where each $f_i, \tilde{f}_i$ are distinct monic irreducible polynomials over $\mathbb{F}_{q^2}$ of degree $k_i$ and each $g_i$ is a distinct monic irreducible polynomial over
\( \mathbb{F}_{q^2} \) of degree \( 2m_i - 1 \). Then

\[
C_{\text{GU}_n(q)}(s) \cong \text{GL}_{a_1}(q^{2k_1}) \times \text{GL}_{a_2}(q^{2k_2}) \times \cdots \times \text{GL}_{a_t}(q^{2k_t}) \times \\
\text{GU}_{b_1}(q^{2m_1 - 1}) \times \text{GU}_{b_2}(q^{2m_2 - 1}) \times \cdots \times \text{GU}_{b_r}(q^{2m_r - 1}),
\]

where \( n = \sum b_i(2m_i - 1) + \sum 2a_i k_i \).

Remark 2.4. The number of \( \text{GL}_{a_i}(q^{2k}) \) appearing in the product is at most one half of the number of monic irreducible polynomials \( f \) over \( \mathbb{F}_{q^2} \) of degree \( d \) such that \( f \neq \tilde{f} \) and the number of \( \text{GU}_{b_i}(q^{2m_i - 1}) \) appearing in the product is at most the number of self-conjugate monic irreducible polynomials over \( \mathbb{F}_{q^2} \) of degree \( 2m - 1 \).

### 3. The General Linear Groups

First, we determine unipotent classes of small sizes of \( \text{GL}_n(q) \).

**Lemma 3.1.** Let \( u \) be a unipotent element of \( \text{GL}_n(q) \) with \( n \geq 5 \). Then one of the following holds:

(i) \(|\text{cl}(u)| = 1 \) (one class),

(ii) \(|\text{cl}(u)| = (q^{n-1} - 1)(q^n - 1)/(q - 1) \) (one class),

(iii) \(|\text{cl}(u)| > q^{4n-8} \).

**Proof.** As mentioned above, since \( u \) is unipotent, \( \lambda_u(f) \) is empty for all \( f \neq x - 1 \). Suppose that \( \lambda_u(x - 1) = \alpha = (\alpha_1, \alpha_2, ...) \), a partition of size \( n \). Formula (2.1) gives us the size of \( C_G(u) \):

\[
|C_G(u)| = q^{\sum (\alpha_i^2)} \prod_i (1/q)_{m_i(\alpha)} = h(q, \alpha).
\]

Remark that the partitions \((1^n)\) and \((2, 1^{n-2})\) correspond respectively to the conjugacy class of the identity element of size 1 and a conjugacy class of size

\[
\frac{|\text{GL}_n(q)|}{h(q, (2, 1^{n-2}))} = \frac{(q^{n-1} - 1)(q^n - 1)}{q - 1}.
\]

If \( \alpha \) is different from \((1^n)\) and \((2, 1^{n-2})\), then Lemma 2.1 implies that

\[
\frac{|\text{GL}_n(q)|}{h(q, \alpha)} \geq \frac{|\text{GL}_n(q)|}{h(q, (2^2, 1^{n-4}))} = \frac{q^{n(n-1)/2}(q - 1)(q^2 - 1) \cdots (q^n - 1)}{q^{(n-2)^2 + 2^2(1/q)_{n-4}(1/q)_{2}}} = \frac{q(q^{n-3} - 1)(q^{n-2} - 1)(q^{n-1} - 1)(q^n - 1)}{(q - 1)(q^2 - 1)} > q^{4n-8}
\]

for every \( n \geq 5 \), as desired. \qed
Remark 3.2. 1) From the correspondence between Jordan canonical forms and partition-valued functions, we see that the Jordan canonical form associated to the conjugacy class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\) in Lemma 3.1 is
\[
\text{diag}(J(2, 1), J(1, 1), \ldots, J(1, 1)).
\]

2) By [8] Lemma 6.1, \(h(q, \alpha) \geq q^{\lfloor \alpha \rfloor}(1 - 1/q)\). Moreover, following the proof, we see that the equality happens if and only if \(\alpha = \lfloor \alpha \rfloor\). Therefore, \(\text{GL}_n(q)\) has a unique conjugacy classes of unipotent elements of maximal size
\[
\frac{\text{GL}_n(q)}{h(q, (n))} = q^{(n-1)^2(q^2 - 1)(q^3 - 1) \cdots (q^n - 1)}.
\]

We now determine semisimple classes of small sizes of \(\text{GL}_n(q)\).

Lemma 3.3. Let \(s\) be a semisimple element of \(G := \text{GL}_n(q)\) with \(n \geq 6\). Then one of the following holds:

(i) \(|\text{cl}(s)| = 1 (q - 1 \text{ classes})\),

(ii) \(|\text{cl}(s)| = q^{n-1}(q^n - 1)/(q - 1) (q^2 - 3q + 2 \text{ classes})\),

(iii) \(|\text{cl}(s)| > q^{4n-8}\).

Proof. The structure of centralizers of semisimple elements in the classical groups is described in [6] and in more detailed in [17]. If the characteristic polynomial of \(s\) is a product \(\prod_{i=1}^{t} f_i^{a_i}(x)\), where each \(f_i\) is a distinct monic irreducible polynomial over \(\mathbb{F}_q\) of degree \(k_i\), then
\[
C_G(s) \cong \text{GL}_{a_1}(q^{k_1}) \times \text{GL}_{a_2}(q^{k_2}) \times \cdots \times \text{GL}_{a_t}(q^{k_t}),
\]
where \(\sum_{i=1}^{t} a_i k_i = n\) and the number of \(\text{GL}_{a_i}(q^{k_i})\) appearing in the product is at most the number of monic irreducible polynomials over \(\mathbb{F}_q\) of degree \(k\).

1) Case \(t = 1\): We know that if \(C_G(s) = \text{GL}_n(q)\) then \(s \in Z(G)\) and hence \(|\text{cl}(s)| = 1\). Since \(|Z(G)| = q - 1\), \(G\) has \(q - 1\) central classes of size 1. Now we can assume that \(C_G(s) = \text{GL}_a(q^k)\) with \(ak = n\) and \(k \geq 2\). In this case, we have
\[
|\text{cl}(s)| = \frac{|\text{GL}_n(q)|}{|\text{GL}_a(q^k)|} = \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - 1)}{q^{ka(a-1)/2} \prod_{j=1}^{a} (q^{kj} - 1)} > \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - 1)}{q^{ka^2}}.
\]

Observe that, as \(q \geq 2\),
\[
\prod_{j=1}^{n} (q^j - 1) = \prod_{j=1}^{n} q^j \left(1 - \frac{1}{q^j}\right) = q^{n(n+1)/2} \prod_{j=1}^{n} \left(1 - \frac{1}{q^j}\right) > q^{n(n+1)/2} \prod_{j=1}^{\infty} \left(1 - \frac{1}{q^j}\right)
\]
Therefore, (3.3) implies that
\[ n^2 \text{ and } GL_a \]
Since polynomial of \( s \) yields
\[ q = n^{(n-1)/2}q^{n(n+1)/2}/2 - 2 \]
This last inequality comes from the observation that 2
\[ n \]
as desired. □

2) Case \( t = 2 \): First we consider the subcase \( C_G(s) = GL_1(q) \times GL_{n-1}(q) \). This happens if and only if \( q \geq 3 \) and the characteristic polynomial of \( s \) is \((x - \lambda_1)(x - \lambda_2)^{n-1}\) for some \( \lambda_1 \neq \lambda_2 \) in \( \mathbb{F}_q \). This yields \( q^2 - 3q + 2 \) different classes of semisimple elements of size \( q^{n-1}(q^2 - 1)/(q - 1)\) and hence (ii) holds.

Next we assume \( C_G(s) \cong GL_1(q) \times GL_a(q^k) \), where \( k \geq 2 \) and \( ak = n - 1 \). We then have
\[
|cl(s)| = \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - 1)}{(q - 1)^a \prod_{j=1}^{ak} (q^{kj} - 1)} > \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - 1)}{q^{ka^2+1}} > \frac{q^{n^2-2}}{q^{(n-1)^2/2+1}} > q^{4n-8}
\]
for every \( n \).

Finally, we consider the subcase \( C_G(s) \cong GL_{a_1}(q^{k_1}) \times GL_{a_2}(q^{k_2}) \) where \( a_1k_1 \) and \( a_2k_2 \) are greater than 1. Since \( |GL_{a_1}(q^{k_1})| \leq |GL_{a_2}(q^{k_2})| \) for \( i = 1, 2 \), we obtain \( |C_G(s)| \leq |GL_{n_1}(q) \times GL_{n_2}(q)| \) for some integers \( n_1 \) and \( n_2 \) such that \( n_1 + n_2 = n \) and \( 1 \notin \{n_1, n_2\} \). Then
\[
|cl(s)| \geq \frac{|GL_{n}(q)|}{|GL_{n_1}(q)||GL_{n_2}(q)|} = \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})q^{n_1(n-n_1)}}{|GL_{n_1}(q)|} = \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-(n_1-1)} - 1)q^{n_1(n-n_1)}}{(q^{n_1} - 1)(q^{n_1-1} - 1)\cdots(q - 1)} > q^{n_1(n-n_1)}q^{n_1(n-n_1)} = q^{2n_1(n-n_1)} \geq q^{4n-8}.
\]
This last inequality comes from the observation that \( 2n_1(n-n_1) \) is minimized when \( n_1 = 2 \) or \( n_1 = n - 2 \).

3) Case \( t \geq 3 \): Choose a subset \( T \subseteq \{1, 2, \cdots, t\} \) so that \( \sum_{i \in T} a_i k_i \geq 2 \) and \( \sum_{i \notin T} a_i k_i \geq 2 \). We then have
\[
|cl(s)| = \frac{|GL_{n}(q)|}{\prod_{i \in T} |GL_{a_i}(q^{k_i})| \prod_{i \notin T} |GL_{a_i}(q^{k_i})|} \geq \frac{|GL_{n}(q)|}{|GL_{\sum_{i \in T} a_i k_i}(q)||GL_{\sum_{i \notin T} a_i k_i}(q)|}.
\]
As proved in the previous case, the last fraction is greater than \( q^{4n-8} \), as desired. □
We are now ready to prove the first main theorem.

**Proof of Theorem 1.1** Let \( g \in G := \text{GL}_n(q) \) and let \( g = su = us \) where \( s \in G \) is semisimple and \( u \in G \) is unipotent be the Jordan decomposition of \( g \). Since \( C_G(g) = C_G(s) \cap C_G(u) \), we get

\[
|C_G(g)| \leq \min\{|C_G(s)|, |C_G(u)|\}.
\]

Assuming that \( |\text{cl}(g)| \leq q^{4n-8} \), then we have \( |\text{cl}(s)| \leq q^{4n-8} \). From Lemma 3.3, we consider two cases:

1) \( |\text{cl}(s)| = 1 \). Then \( s \) is one of the \( q - 1 \) central elements in \( G \). Therefore, \( |\text{cl}(u)| = |\text{cl}(g)| \leq q^{4n-8} \). By Lemma 3.1, \( u \) either is identity or belongs to the unipotent class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\). Now (i) holds in the former case and (ii) holds in the latter case.

2) \( |\text{cl}(s)| = q^{n-1}(q^n - 1)/(q - 1) \). Then \( q \geq 3 \) and, in particular, \( C_G(s) \cong \text{GL}_1(q) \times \text{GL}_{n-1}(q) \). If \( u = 1 \) then we get \( q^2 - 3q + 2 \) conjugacy classes of semisimple elements and hence (iii) holds. So we can assume \( u \neq 1 \).

Since \( u \) commutes with \( s \), we can consider \( u \) as a unipotent element of \( C_G(s) \cong \text{GL}_1(q) \times \text{GL}_{n-1}(q) \). Using Lemma 3.1, we have

\[
|\text{cl}_{C_G(s)}(u)| \geq \frac{(q^{n-2} - 1)(q^{n-1} - 1)}{q - 1}.
\]

It follows that

\[
|\text{cl}(g)| = \frac{|\text{GL}_n(q)|}{|C_G(g)|} = \frac{|\text{GL}_n(q)|}{|C_G(s)|} \cdot \frac{|C_G(s)|}{|C_G(s)|} = |\text{cl}(s)| \cdot |\text{cl}_{C_G(s)}(u)|
\]

\[
\geq \frac{q^{n-1}(q^n - 1)}{q - 1} \cdot \frac{(q^{n-2} - 1)(q^{n-1} - 1)}{q - 1} > q^{4n-8}
\]

for every \( n \geq 6 \), as claimed. \( \square \)

### 4. Other linear groups

In this section, we derive from Theorem 1.1 some corresponding gap results on conjugacy class sizes of \( \text{SL}_n(q) \), \( \text{PGL}_n(q) \), and \( \text{PSL}_n(q) \).

#### 4.1. The special linear groups.

**Corollary 4.1.** Let \( n \geq 6 \). Then \( \text{SL}_n(q) \) has \( \gcd(n, q - 1) \) (central) classes of size 1, \( \gcd(n, q - 1) \) classes of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\), \( q - 1 - \gcd(n, q - 1) \) (semisimple) classes of size \( q^{n-1}(q^n - 1)/(q - 1) \) and all other classes have sizes greater than \( q^{4n-9} \).
Proof. Let \( G := \text{GL}_n(q) \) and \( S := \text{SL}_n(q) \). It is clear that \(|Z(S)| = \gcd(n, q-1)\) and therefore \( S \) has exactly \( \gcd(n, q-1) \) classes of size 1. Let \( g \in S \leq G \) and \( g \) is not central in \( G \).

Suppose that \( g \) belongs to a \( G \)-conjugacy class of size \((q^n-1)/q-1)\). From the proof of Theorem 1.1, we know that \( g = au \), where \( a \in \mathbb{F}_q^\times \) and \( u \) is unipotent with \( \lambda_u(x-1) = (2, 1^{n-2}) \) and \( \lambda_u(f) \) is empty for every \( f \neq x-1 \). Without loss, assume that \( u = \text{diag}(J(2, 1), J(1, 1), \ldots, J(1, 1)) \). Define

\[
\varphi : C_G(u) \rightarrow \mathbb{F}_q^\times \quad x \mapsto \det(x).
\]

For each \( k \in \mathbb{F}_q^\times \), the matrix \( \text{diag}(1, 1, k, 1, \ldots, 1) \) commutes with \( u \) has determinant \( k \). Therefore \( \varphi \) is surjective and it follows that \(|C_G(u)| = (q-1)|C_S(u)|\). Thus, \(|\text{cl}_G(u)| = (q^n-1)/(q^n-1)\). We note that there are \( \gcd(n, q-1) \) choices for \( a \) and, hence, \( \gcd(n, q-1) \) classes of \( G \)-conjugacy class of size \((q^n-1)/(q^n-1)\).

Next we suppose that \( g \) belongs to a \( G \)-conjugacy class of size \((q^n-1)/(q^n-1)\). Then \( q \geq 3 \) and \( C_G(g) \cong \text{GL}_1(q) \times \text{GL}_{n-1}(q) \). Thus, we see that \( b_k := \left( \begin{smallmatrix} k & 0 \\ 0 & I_{n-1} \end{smallmatrix} \right) \in C_G(g) \) for each \( k \in \mathbb{F}_q^\times \) and \( \det(b_k) = k \). Utilizing the determinant map once again, we obtain \(|C_G(g)| = (q-1)|C_S(g)|\) and \(|\text{cl}_G(g)| = |\text{cl}_S(g)| = q^{n-1}/(q^n-1)\). Recall that the characteristic polynomial of \( g \) is \((x - \lambda_1)(x - \lambda_2)^{n-1}\) for some \( \lambda_1 \neq \lambda_2 \in \mathbb{F}_q^\times \).

We have \( \lambda_1\lambda_2^{n-1} = 1 \) and \( \lambda_2^n \neq 1 \) as \( g \) is not central. Thus there are \( q-1 - \gcd(n, q-1) \) choices for \( \lambda_2 \), leaving only one choice for \( \lambda_1 \), giving \( q-1 - \gcd(n, q-1) \) conjugacy classes of \( S \) of size \((q^n-1)/(q^n-1)\).

Finally, if \( g \) belongs to a \( G \)-conjugacy class of size greater than \( q^{4n-8} \) then

\[
|\text{cl}_G(g)| = |G|/|C_G(g)| \leq (q-1)|S|/|C_S(g)| = (q-1)|\text{cl}_S(g)|.
\]

Hence

\[
|\text{cl}_S(g)| \geq |\text{cl}_G(g)|/(q-1) \geq q^{4n-8}/(q-1) > q^{4n-9},
\]

as desired. \( \square \)

4.2. The projective linear groups.

**Corollary 4.2.** Let \( n \geq 6 \). Then \( \text{PGL}_n(q) \) has one (central) class of size 1, one (unipotent) class of size \((q^n-1)(q^n-1)/(q-1)\), \( q-2 \) (semisimple) classes of size \((q^n-1)(q^n-1)/(q-1)\) and all other classes have sizes greater than \( q^{4n-8}/\gcd(n, q-1) \).

**Proof.** Let \( G := \text{GL}_n(q), Z := Z(G), \) and \( \overline{G} := \text{PGL}_n(q) = G/Z \). For each \( \overline{g} = gZ \in \overline{G} \), let \( C_{\overline{g}} \) denote the inverse image of \( C_{\overline{G}}(\overline{g}) \) under the canonical homomorphism from \( G \) to \( \overline{G} \). For each \( h \in C_{\overline{g}} \), we have
\( T_g : C_g \rightarrow Z \)

\[ h \mapsto z_h = h^{-1}g^{-1}hg. \]

It is easy to see that \( T_g \) is a homomorphism and \( z_h = 1 \) if and only if \( h \in C_G(g) \). Therefore,

\[ \frac{C_g}{C_G(g)} \cong \text{Im}(T_g) \leq Z. \]

Suppose that \( g \) belongs to a \( G \)-conjugacy class of size \( (q^{n-1} - 1)(q^n - 1)/(q - 1) \). Then \( g = ku \), where \( u \) is unipotent with \( |\text{cl}_G(u)| = (q^{n-1} - 1)(q^n - 1)/(q - 1) \) and \( k \in \mathbb{F}_q^\times \). Now let \( h \in C_g \) and suppose \( T_g(h) = z_h = a \cdot 1 \). Then \( hg = (gh)z_h \) implies \( huh^{-1} = uz_h = au \). Taking eigenvalues of both sides, we see that the eigenvalues of \( huh^{-1} \) are all 1; whereas, the eigenvalues of \( au \) are all \( a \). Thus, we must have \( a = 1 \). Hence \( \text{Im}(T_g) = 1 \). It then follows that

\[ |\text{cl}_G(g)| = (q^{n-1} - 1)(q^n - 1)/(q - 1). \]

Since an element from each \( G \)-conjugacy class of size \( (q^{n-1} - 1)(q^n - 1)/(q - 1) \) is contained in \( \mathcal{G} \), elements of this form contribute one \( \mathcal{G} \)-conjugacy class of size \( (q^{n-1} - 1)(q^n - 1)/(q - 1) \).

Next we suppose that \( g \) belongs to a \( G \)-conjugacy class of size \( q^{n-1}(q^n - 1)/(q - 1) \). The characteristic polynomial of \( g \) is \((x - \lambda_1)(x - \lambda_2)^{n-1}\) for some \( \lambda_1 \neq \lambda_2 \in \mathbb{F}_q^\times \). Now let \( h \in C_g \) and suppose \( T_g(h) = z_h = a \cdot 1 \). Then \( hg = (gh)z_h \), from which it follows that \( hgh^{-1} = gz_h = ag \). Taking eigenvalues of both sides, we see that the eigenvalues of \( hgh^{-1} \) are \( \lambda_1 \) (with multiplicity 1) and \( \lambda_2 \) (with multiplicity \( n - 1 \)); whereas, the eigenvalues of \( ag \) are \( a\lambda_1 \) (with multiplicity 1) and \( a\lambda_2 \) (with multiplicity \( n - 1 \)). Since \( n \neq 2 \), the only way this may be is that \( a = 1 \). Hence \( \text{Im}(T_g) = 1 \) and

\[ |\text{cl}_G(g)| = q^{n-1}(q^n - 1)/(q - 1). \]

Since each scalar multiple of \( g \) has the same \( G \)-conjugacy class size as \( g \) and cannot be conjugate to \( g \), the coset \( \mathcal{G} \) contains \( q - 1 \) elements of different conjugacy classes of size \( q^{n-1}(q^n - 1)/(q - 1) \). Thus elements of this form contribute \( (q^2 - 3q + 2)/(q - 1) = q - 2 \) conjugacy classes of \( \mathcal{G} \) of size \( q^{n-1}(q^n - 1)/(q - 1) \).

Finally, suppose that \( g \) belongs to a \( G \)-conjugacy class of size greater than \( q^{n-8} \). Let \( h \in C_g \) and suppose \( T_g(h) = z_h \). It follows that \( gh = (hg)z_h \) and, by taking determinants of both sides, we see that
det \( z_h = 1 \). It follows that \( |\text{Im}(T_g)| \leq \gcd(n, q - 1) \). Therefore \((q - 1)|C_G(\overline{g})| \leq |C_G(g)| \gcd(n, q - 1) \), whence it follows that

\[
\frac{|G|}{|C_G(g)|} \leq \frac{|G|}{(q - 1)|C_G(\overline{g})|} \gcd(n, q - 1).
\]

Thus \( \text{cl}_{\overline{g}}(g) \) \( \geq |\text{cl}_G(g)|/\gcd(n, q - 1) > q^{4n - 8}/\gcd(n, q - 1) \), and claimed.

4.3. The projective special linear groups.

**Corollary 4.3.** Let \( n \geq 6 \). Then \( \text{PSL}_n(q) \) has one (central) class of size 1, one (unipotent) class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\), \((q-1)/\gcd(n, q-1) - 1\) (semisimple) classes of size \(q^{n-1}(q^n - 1)/(q - 1)\) and all other classes have sizes greater than \(q^{4n-9}/\gcd(n, q - 1)\).

**Proof.** Let \( S := \text{SL}_n(q) \), \( Z := Z(S) \), and let \( \overline{S} := \text{PSL}_n(q) = S/Z \). Let \( \varphi \) be the canonical homomorphism from \( S \) to \( \overline{S} \) and let \( \overline{g} = gZ \).

Also define \( C_g \) and the homomorphism \( T_g \) analogously as in the proof of Corollary 4.2. It is easy to see that \( z_h = 1 \) if and only if \( h \in C_S(g) \).

Therefore

\[
\frac{C_g}{C_S(g)} \cong \text{Im}(T_g) \leq Z
\]

Suppose that \( g \) belongs to an \( S \)-conjugacy class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\). Then \( g = ku \), where \( u \) is unipotent with \( |\text{cl}_S(u)| = (q^{n-1} - 1)(q^n - 1)/(q - 1) \) and \( k \in \mathbb{F}_q^\times \) such that \( k^n = 1 \). Using an eigenvalue argument similar to the one used in the proof of Corollary 4.2, we obtain that \( \text{Im}(T_g) = 1 \). Therefore

\[
|\text{cl}_{\overline{S}}(\overline{g})| = |\text{cl}_S(g)| = (q^{n-1} - 1)(q^n - 1)/(q - 1).
\]

Since an element from each \( S \)-conjugacy class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\) is contained in \( \overline{g} \), elements of this form contribute one \( \overline{S} \)-conjugacy class of size \((q^{n-1} - 1)(q^n - 1)/(q - 1)\).

Now suppose that \( g \) belongs to an \( S \)-conjugacy class of size \( q^{n-1}(q^n - 1)/(q - 1) \). Using the eigenvalue argument again, we have

\[
|\text{cl}_S(g)| = |\text{cl}_{\overline{S}}(\overline{g})| = q^{n-1}(q^n - 1)/(q - 1).
\]

Since each scalar multiple \( k \cdot g \) of \( g \) with \( k^n = 1 \) has the same \( S \)-conjugacy class size as \( g \) and cannot be conjugate to \( g \), the coset \( \overline{g} \) contains \( \gcd(n, q - 1) \) elements of different conjugacy classes of size \( q^{n-1}(q^n - 1)/(q - 1) \). Thus elements of this form contribute \((q - 1 - \gcd(n, q - 1))/\gcd(n, q - 1) = (q - 1)/\gcd(n, q - 1) - 1\) conjugacy classes of \( \overline{S} \) of size \( q^{n-1}(q^n - 1)/(q - 1) \).
Now suppose that \( g \) belongs to an \( S \)-conjugacy class of size greater than \( q^{4n-9} \). Since \( |C_{S}(\mathcal{F})| \leq |C_{S}(g)| \), it follows that
\[
|\text{cl}(\mathcal{F})| \geq |\text{cl}(g)|/ \gcd(n, q - 1) > q^{4n-9}/ \gcd(n, q - 1),
\]
and we are done. \( \square \)

5. The General Unitary Groups

For the remainder of the paper, we let \( K := \{ k \in \mathbb{F}_{q}^{\times} | k^{q+1} = 1 \} \). The unipotent and semisimple classes of small size of \( \text{GU}_{n}(q) \) are determined in the next two lemmas.

**Lemma 5.1.** Let \( u \) be a unipotent element of \( G := \text{GU}_{n}(q) \) with \( n \geq 4 \). Then one of the following holds

(i) \( |\text{cl}(u)| = 1 \) (one class),
(ii) \( |\text{cl}(u)| = (q^{n} - (-1)^{n})(q^{n-1} - (-1)^{n-1})/(q + 1) \) (one class)
(iii) \( |\text{cl}(u)| > q^{4n-9} \).

**Proof.** Similar to the proof of Lemma 3.1 \( \square \)

**Lemma 5.2.** Let \( s \) be a semisimple element of \( G := \text{GU}_{n}(q) \) with \( n \geq 5 \). Then one of the following holds:

(i) \( |\text{cl}(s)| = 1 \) \( (q + 1) \) classes,
(ii) \( |\text{cl}(s)| = q^{n-1}(q^{n} - (-1)^{n})/(q + 1) \) \( (q^{2} + q) \) classes,
(iii) \( |\text{cl}(s)| > q^{4n-9} \).

**Proof.** Recall from Lemma 2.3 that
\[
C_{G}(s) \cong \bigotimes_{i=1}^{t} \text{GL}_{a_{i}}(q^{2k_{i}}) \times \bigotimes_{i=1}^{r} \text{GU}_{b_{i}}(q^{2m_{i} - 1}),
\]
where \( \sum a_{i}k_{i} + \sum m_{i}b_{i}(2m_{i} - 1) = n \).

1) **Case \( t=0, r=1 \):** Remark that \( C_{G}(s) \cong \text{GU}_{n}(q) \) if and only if \( s \in Z(G) \) and in that case \( |\text{cl}(s)| = 1 \). Since \( |Z(G)| = q + 1 \), the group \( G \) has \( q + 1 \) central classes of size \( 1 \) and (i) holds. If \( C_{G}(s) \cong \text{GU}_{b}(q^{2m-1}) \) with \( b(2m - 1) = n \) and \( m \geq 2 \), then we have for every \( n \geq 5 \),
\[
|\text{cl}(s)| = q^{n(n-1)/2} \prod_{j=1}^{n} (q^{j} - (-1)^{j})^{b_{j}} \prod_{j=1}^{b} (q^{2m_{j}-1} - (-1)^{j})^{b_{j}} > q^{n(n-1)/2} \prod_{j=1}^{n} (q^{j} - 1) \prod_{j=1}^{b} (q^{2m_{j}-1} + 1) = q^{n^{2} - 2} \prod_{j=1}^{b} (q^{2m_{j}} + 1) = q^{n^{2} - 4b} > q^{4n-9}.
\]

2) **Case \( t=0, r=2 \):** First we consider the subcase \( C_{G}(s) \cong \text{GU}_{1}(q) \times \text{GU}_{n-1}(q) \). This happens if and only if the characteristic polynomial of
s is \((x - \lambda_1)(x - \lambda_2)^{n-1}\) where \(\lambda_1, \lambda_2 \in K\) and \(\lambda_1 \neq \lambda_2\). Thus there are \(q^2 + q\) different classes of size

\[
|\text{cl}(s)| = \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - (-1)^j)}{(q + 1)q^{(n-1)(n-2)/2} \prod_{j=1}^{n-1} (q^j - (-1)^j)} = \frac{q^{n-1}q^n - (-1)^n}{q + 1}
\]

and (ii) holds.

Next, we suppose that \(C_G(s) \cong \text{GU}_1(q) \times \text{GU}_b(q^{2m-1})\), where \(b(2m - 1) = n - 1\) and \(m \geq 2\). In this case, we have

\[
|\text{cl}(s)| = \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - (-1)^j)}{(q + 1)q^{(n-1)(b-1)/2} \prod_{j=1}^{b} (q^{2m-1}j - (-1)^j)}
\]

\[
\geq \frac{q^{n(n-1)/2} \prod_{j=1}^{n} (q^j - 1)}{(q + 1)q^{(n-1)(b-1)/2} \prod_{j=1}^{b} (q^{2m-1}j + 1)}
\]

\[
> \frac{q^{n(n-1)/2}q^{n+1/2}}{(q + 1)q^{n(b+1)/2}q^{2m-1}b(b+1)+2}
\]

\[
= \frac{q^{n^2-2}}{(q + 1)q^{n(n-1)(b-1)/2}q^{n(b+1)+2}}
\]

\[
= \frac{q^{n^2-(n-1)b-4}}{(q + 1)} > q^{n^2-(n-1)^2/3-6} \geq q^{4n-9}.
\]

Finally, we consider the subcase \(C_G(s) \cong \text{GU}_{b_1}(q^{2m_1-1}) \times \text{GU}_{b_2}(q^{2m_2-1})\) where \(b_1(2m_1 - 1) \geq 2\) and \(b_2(2m_2 - 1) \geq 2\). Since \(|\text{GU}_{b_i}(q^{2m_i-1})| \leq |\text{GU}_{b_i(2m_i-1)}(q)|\) for \(i = 1, 2\), we obtain \(|C_G(s)| \leq |\text{GU}_{n_1}(q) \times \text{GU}_{n_2}(q)|\)

for some positive integers \(n_1\) and \(n_2\) greater than 1 such that \(n_1 + n_2 = n\). It follows that

\[
|\text{cl}(s)| \geq \frac{|\text{GU}_n(q)|}{|\text{GU}_{n_1}(q)||\text{GU}_{n_2}(q)|} = q^{n_1(n-n_1)} \prod_{j=1}^{n_1} (q^j - (-1)^j)
\]

\[
= q^{n_1(n-n_1)} \frac{\prod_{j=n-n_1+1}^{n_1} (q^j - (-1)^j)}{\prod_{j=1}^{n_2} (q^j - (-1)^j)} = q^{2n_1(n-n_1)} \frac{\prod_{j=n-n_1+1}^{n} (1 + (1/q)j)}{\prod_{j=1}^{n_1-1} (1 + (1/q)j)}
\]

\[
\geq q^{2n_1(n-n_1)} \left[ \frac{1 - 1/q^{n_1(n-1)}}{1 + 1/q} \right]^{n_1/2} \geq q^{2n_1(n-n_1)} \left[ \frac{1 - 1/q^{n_1(n-1)}}{1 + 1/q} \right]^{n_1/2}
\]

\[
= q^{n_1(n-n_1)} \left[ \frac{q^{n(n-1)-1} + 1}{q + 1} \right]^{n_1/2} \geq q^{2n_1(n-n_1)} \frac{q}{q^{n(n-1)-1} + 1}
\]

\[
> q^{2n_1(n-n_1)} \geq q^{4n-8} > q^{4n-9}.
\]

3) Case \(t=0, r \geq 3\): Then \(C_G(s) = \bigotimes_{i=1}^{r} \text{GU}_{b_i}(q^{2m_i-1})\) where \(r \geq 3\) and \(\sum b_i(2m_i - 1) = n\). Choose a subset \(T \subseteq \{1, 2, \ldots, r\}\) so that
\[
\sum_{i \in T} b_i(2m_i - 1) \geq 2 \quad \text{and} \quad \sum_{i \in T} b_i(2m_i - 1) \geq 2.
\]

Then we have
\[
|\text{cl}(s)| = \frac{|\text{GU}_n(q)|}{\prod_{i \in T}|\text{GU}_{b_i}(q^{2m_i-1})|} \times \frac{|\text{GU}_n(q)|}{\prod_{i \in T}|\text{GU}_{b_i}(q^{2m_i-1})|} \\
\geq \frac{|\text{GU}_{\sum_{i \in T} b_i(2m_i-1)}(q)|}{|\text{GU}_{\sum_{i \in T} b_i(2m_i-1)}(q)|} > q^{4n-9}
\]
as was proved in the previous case.

4) Case \( t \geq 1, r \geq 0 \): First we assume that \( r = 0 \). Then \( n \) must be even and we have
\[
|\text{cl}(s)| = \frac{|\text{GU}_n(q)|}{\prod_{i=1}^t |\text{GL}_{a_i}(q^{2k_i})|} > \frac{|\text{GU}_n(q)|}{|\text{GL}_{n/2}(q^2)|} = \frac{q^{n/2} \prod_{j=1}^n (q^j - (-1)^j)}{\prod_{j=1}^{n/2} (q^{2j} - 1)} \\
\geq \frac{q^{n/2} \prod_{j=1}^n (q^j - (-1)^j)}{q^{n(n+2)/8}} > \frac{q^{n/2} q^{n(n+1)/2-2}}{q^{n(n+2)/8}} \\
= q^{3n^2/8+3n/4-2} > q^{4n-9}.
\]

Next, suppose that \( r \geq 1 \). If \( \sum_{i=1}^r b_i(2m_i - 1) = 1 \), then \( n \) is odd and
\[
|\text{cl}(s)| = \frac{|\text{GU}_n(q)|}{|\text{GU}_1(q)| \prod_{i=1}^r |\text{GL}_{a_i}(q^{2k_i})|} \times \frac{|\text{GU}_n(q)|}{|\text{GL}_{n/2}(q^2)|} \\
\geq \frac{q^{3(n-1)/2} \prod_{j=1}^n (q^j - (-1)^j)}{(q+1) \prod_{j=1}^{n-1/2} (q^{2j} - 1)} > \frac{q^{3(n-1)/2} \prod_{j=1}^n (q^j - (-1)^j)}{(q+1) q^{n(n-1)(n+1)/4}} \\
\geq \frac{q^{3(n-1)/2} q^{n(n+1)/2-2}}{(q+1) q^{n(n-1)(n+1)/4}} > q^{n^2/4+2n-21/4} \geq q^{4n-9}.
\]

On the other hand, if \( \sum_{i=1}^r b_i(2m_i - 1) \geq 2 \), then
\[
|\text{cl}(s)| = \frac{|\text{GU}_n(q)|}{\prod_{i=1}^r |\text{GL}_{a_i}(q^{2k_i})| \prod_{i=1}^r |\text{GU}_{b_i}(q^{2m_i-1})|} \\
\geq \frac{|\text{GU}_n(q)|}{\prod_{i=1}^r |\text{GU}_{b_i}(q^{2m_i-1})|} \\
\geq \frac{|\text{GU}_n(q)|}{|\text{GU}_{\sum_{i=1}^r 2a_i k_i}(q)| |\text{GU}_{\sum_{i=1}^r b_i(2m_i-1)}(q)|} > q^{4n-9}
\]
as was proved in the last subcase of 2). \( \square \)

Proof of Theorem 1.2 Let \( g \in G := \text{GU}_n(q) \) and let \( g = su = us \) where \( s \in G \) is semisimple and \( u \in G \) is unipotent be the Jordan decomposition of \( g \). Since \( C_G(g) = C_G(s) \cap C_G(u) \), we get
\[
|C_G(g)| \leq \min\{|C_G(s)|, |C_G(u)|\}.
\]
Assuming that $|\text{cl}(g)| \leq q^{4n-9}$, then we have $|\text{cl}(s)| \leq q^{4n-9}$. From Lemma 5.2, we have two cases:

1) $|\text{cl}(s)| = 1$. Then $s$ is one of the $q+1$ central elements in $G$. Therefore, $|\text{cl}(u)| = |\text{cl}(g)| \leq q^{4n-9}$. By Lemma 5.1, $u$ either is identity or belongs to a class of size $(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)/(q + 1)$. Now (i) holds in the former case and (ii) holds in the latter case.

2) $|\text{cl}(s)| = q^{n-1}(q^n - (-1)^n)/(q + 1)$. Then $C_G(s) \cong GU_1(q) \times GU_{n-1}(q)$. If $u = 1$ then we get $q^2 + q$ conjugacy classes of semisimple elements and hence (iii) holds. So we can assume $u \neq 1$.

Since $u$ commutes with $s$, we can consider $u$ as a unipotent element of $C_G(s)$. Using Lemma 5.1 we have

$$|\text{cl}_{C_G(s)}(u)| \geq \frac{(q^{n-2} - (-1)^{n-2})(q^{n-1} - (-1)^{n-1})}{q + 1}.$$  

It follows that

$$|\text{cl}(g)| = |\text{cl}(s)| \cdot |\text{cl}_{C_G(s)}(u)|$$

$$\geq \frac{q^{n-1}(q^n - (-1)^n)}{q + 1}, \frac{(q^{n-2} - (-1)^{n-2})(q^{n-1} - (-1)^{n-1})}{q + 1}$$

$$> \frac{q^{n-1}(q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1)}{(q + 1)^2} > q^{4n-9},$$

as claimed. \qed

6. Other unitary groups

As for the linear groups, we can derive from Theorem 1.2 some gap results on conjugacy class sizes for $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q)$. We leave the proofs in this section to the reader as they are similar to those in Section 4.

**Corollary 6.1.** Let $n \geq 5$. Then $SU_n(q)$ has $\gcd(n, q + 1)$ (central) classes of size 1, $\gcd(n, q + 1)$ classes of size $(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)/(q+1)$, $q+1-\gcd(n, q+1)$ (semisimple) classes of size $q^{n-1}(q^n - (-1)^n)/(q+1)$ and all other classes have sizes greater than $q^{4n-9}/(q+1)$.

**Corollary 6.2.** Let $n \geq 5$. Then $PGU_n(q)$ has one (central) class of size 1, one (unipotent) class of size $(q^{n-1} - (1)^{n-1})(q^n - (-1)^n)/(q+1)$, $q$ (semisimple) classes of size $q^{n-1}(q^n - (1)^n)/(q+1)$ and all other classes have sizes greater than $q^{4n-9}/\gcd(n, q + 1)$.

**Corollary 6.3.** Let $n \geq 5$. Then $PSU_n(q)$ has one (central) class of size 1, one (unipotent) class of size $(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)/(q+1)$, $(q + 1)/\gcd(n, q + 1) - 1$ (semisimple) classes of size $q^{n-1}(q^n - (1)^n)/(q+1)$.
\((-1)^n/(q+1)\) and all other classes have sizes greater than \(q^{4n-9}/((q+1) \gcd(n,q+1))\).

7. Applications: Bounds and limits for the zeta-type function

We apply the results in previous sections to obtain some bounds and limits for the zeta-type function encoding the conjugacy class sizes of the linear and unitary groups. We recall that the function associated to a finite group \(G\) is

\[
\zeta^G(t) = \sum_{C \in \mathcal{C}(G)} |C|^{-t} = \sum_{s \in \mathbb{N}} k_s(G)s^{-t}
\]

where \(k_s(G)\) denotes the number of classes of \(G\) of size \(s\).

Lemma 7.1. Suppose that \(n \geq 6\) and \(t > 0\). Then

\[
\zeta^{\text{GL}_n(q)}(t) < (q - 1) + (q - 1)^2q^{-t(2n-2)} + q^{n-t(4n-8)}.
\]

Proof. Bounds for the number of classes in finite Chevalley groups are determined in \([8]\). In particular, the number of classes of \(\text{GL}_n(q)\) is bounded above by \(q^n\) (cf. Proposition 3.5 of \([8]\)). This and Theorem 1.1 imply that

\[
\zeta^{\text{GL}_n(q)}(t) = (q - 1) + (q - 1)(\frac{q^{n-1} - 1}{q - 1})^{-t} + (q^2 - 3q + 2)(\frac{q^{n-1} - 1}{q - 1})^{-t} + \sum_{s \geq q^{4n-8}} k_s(\text{GL}_n(q))s^{-t}
\]

\[
< (q - 1) + q^{-t(2n-2)}(q - 1) + q^{-t(2n-2)}(q^2 - 3q + 2) + q^nq^{-t(4n-8)}
\]

\[
= (q - 1) + q^{-t(2n-2)}(q - 1)^2 + q^{n-t(4n-8)}.
\]

□

Lemma 7.2. Suppose that \(n \geq 5\) and \(t > 0\). Then

\[
\zeta^{\text{GU}_n(q)}(t) < (q + 1) + (q + 1)^2q^{-t(2n-3)} + (q^n + 16q^{n-1})q^{-t(4n-9)}.
\]
Proof. The number of classes of $\text{GU}_n(q)$ is bounded above by $q^n + 16q^{n-1}$ (cf. Proposition 3.9 of [8]). This and Theorem 1.2 imply that

$$
\zeta^{\text{GU}_n(q)}(t) = (q+1) + (q+1)\left(\frac{(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)}{q+1}\right)^{-t} + (q^2 + q)\left(\frac{q^{n-1}(q^n - (-1)^n)}{q+1}\right)^{-t} + \sum_{s > q^{4n-9}} k_s(\text{GU}_n(q))s^{-t} < (q+1) + q^{-t(2n-3)}(q+1) + q^{-t(2n-3)}(q^2 + q) + (q^n + 16q^{n-1})q^{-t(4n-9)}
$$

$$
= (q+1) + q^{-t(2n-3)}(q+1)^2 + (q^n + 16q^{n-1})q^{-t(4n-9)}.
$$

□

Theorem 1.3 now is just a consequence of Theorems 7.1 and 7.2.

Proof of Theorem 1.3. Recall the hypothesis that $n \geq 6$ and $t > 1/3$. Lemma 7.1 implies that

$$
\zeta^{\text{GL}_n(q)}(t) < \left(1 - \frac{1}{q}\right) + \left(1 - \frac{2}{q} + \frac{1}{q^2}\right)q^{1-t(2n-2)} + q^{n-1-t(4n-8)}.
$$

As $n \geq 6$ and $t > 1/3$, we have $1-t(2n-2) < 0$ and $n-1-t(4n-8) < 0$. Therefore, the right side of the above inequality tends to 1 uniformly in $n$ as $q \to \infty$. As it is clear that $\zeta^{\text{GL}_n(q)}(t)/q > q-1$, we conclude

$$
\frac{\zeta^{\text{GL}_n(q)}}{q} \to 1 \text{ uniformly in } n \text{ as } q \to \infty.
$$

Similarly, Lemma 7.2 yields

$$
\zeta^{\text{GU}_n(q)}(t) < \left(1 + \frac{1}{q}\right) + \left(1 + \frac{2}{q} + \frac{1}{q^2}\right)q^{1-t(2n-3)} + (q^{n-1} + 16q^{n-2})q^{-t(4n-9)}.
$$

Now the proof proceeds as in the $\text{GL}_n(q)$ case. □

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