Irreducible restrictions of Brauer characters of the Chevalley group $G_2(q)$ to its proper subgroups

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Abstract

Let $G_2(q)$ be the Chevalley group of type $G_2$ defined over a finite field with $q = p^n$ elements, where $p$ is a prime number and $n$ is a positive integer. In this paper, we determine when the restriction of an absolutely irreducible representation of $G$ in characteristic other than $p$ to a maximal subgroup of $G_2(q)$ is still irreducible. Similar results are obtained for $2B_2(q)$ and $2G_2(q)$.

Keywords: Irreducible restrictions; Brauer characters; Chevalley group $G_2(q)$

1. Introduction

Let $G$ be the Chevalley group $G_2(q)$ defined over a finite field with $q = p^n$ elements, where $p$ is a prime number and $n$ is a positive integer. Let $M$ be a maximal subgroup of $G$ and $Φ$ an absolutely irreducible representation of $G$ in cross characteristic $ℓ$ (i.e. $ℓ = 0$ or $ℓ$ prime, $ℓ \nmid q$). Let $ϕ$ be the irreducible Brauer character of $G$ afforded by $Φ$. The purpose of this paper is to find all possibilities of $ϕ$ and $M$ such that $ϕ|_M$ is also irreducible. This result is a contribution to the classification of maximal subgroups of finite classical groups guided by Aschbacher’s theorem [A].

In order to do that, we need to use the results about maximal subgroups, character tables, blocks and Brauer trees of $G$ obtained by many authors.
The list of maximal subgroups of $G$ is determined by Cooperstein in [Cp] for $p = 2$ and Kleidman in [K1] for $p$ odd. The complex character table of $G$ is determined by Chang and Ree in [CR] for good primes $p$ (i.e. $p \geq 5$). Enomoto computed in [E] the character table of $G$ when $p = 3$. Lastly, when $p = 2$, it is computed by Enomoto and Yamada in [EY]. In a series of papers [H1,HS1,HS2,S2,S3,S4], Hiss and Shamash have determined the blocks, Brauer trees and (almost completely) the decomposition numbers for $G$.

In order to solve the problem, it turns out to be useful to know the small degrees of irreducible Brauer characters of $G$. The smallest degrees of nontrivial absolutely irreducible representations of many quasi-simple groups are collected in [T1] and the result for $G$ is used in this paper. From the knowledge of the degrees of irreducible Brauer characters of $G$, we compute the second smallest such degree. The exact formula for it is given in Theorem 3.1. This is also useful for other applications.

Clearly, it suffices to consider maximal subgroups $M$ of $G$ with $\sqrt{|M|}$ larger than the smallest degree of nontrivial absolutely irreducible representations of $G$. Thus, we can exclude many maximal subgroups of $G$ whose orders are small enough by Reduction Theorem 4.4. The remaining maximal subgroups are treated individually by various methods. The main results of this paper are the following:

**Theorem 1.1.** Let $G = G_2(q)$, $q = p^n$, $q \geq 5$, $p$ a prime number. Let $\varphi$ be an absolutely irreducible character of $G$ in cross characteristic $\ell$ and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$. Then $\varphi|M$ is irreducible if and only if one of the following holds:

(i) $q \equiv -1 \pmod{3}$, $M = \text{SL}_3(q) : 2$ and $\varphi$ is the unique character of the smallest degree $q^3 - 1$.
(ii) $q \equiv 1 \pmod{3}$, $M = \text{SU}_3(q) : 2$ and $\varphi$ is the unique character of the smallest degree. In this case, $\varphi(1) = q^3$ when $\ell = 3$ and $\varphi(1) = q^3 + 1$ when $\ell \neq 3$.

The covering groups of $G_2(3)$ and $G_2(4)$ are handled by the following theorem. Notice that $3 \cdot G_2(3)$ has two pairs of complex conjugate irreducible characters of degree 27.

**Theorem 1.2.** Let $G \in \{G_2(3), 3 \cdot G_2(3), G_2(4), 2 \cdot G_2(4)\}$. Let $\varphi$ be a faithful absolutely irreducible character of $G$ in cross characteristic $\ell$ of degree larger than 1 and $M$ a maximal subgroup of $G$. Then we have:

(i) When $G = G_2(3)$, $\varphi|M$ is irreducible if and only if one of the following holds:
   (a) $M = U_3(3) : 2$, $\varphi$ is the unique irreducible character of degree 14 when $\ell = 7$ or $\varphi$ is any of the irreducible characters of degree 14, 64 when $\ell \neq 3, 7$;
   (b) $M = 2^3 \cdot L_3(2)$, $\varphi$ is the unique irreducible character of degree 14 when $\ell \neq 2, 3$.
(ii) When $G = 3 \cdot G_2(3)$, the universal cover of $G_2(3)$, $\varphi|M$ is irreducible if and only if one of the following holds:
   (a) $M = 3.P$ or $3.Q$ where $P$, $Q$ are maximal parabolic subgroups of $G_2(3)$, $\varphi$ is any of the four irreducible characters of degree 27;
   (b) $M = 3.(U_3(3) : 2)$, $\varphi$ is any of the two complex conjugate irreducible characters of degree 27 when $\ell \neq 2, 3, 7$;
   (c) $M = 3.(L_3(3) : 2)$, $\varphi$ is any of the two complex conjugate irreducible characters of degree 27 when $\ell \neq 2, 3, 13$;
   (d) $M = 3.(L_2(8) : 3)$, $\varphi$ is any of the four irreducible characters of degree 27 when $\ell \neq 2, 3, 7$. 
Lemma 2.3. When $G = G_2(4)$, $\varphi|_M$ is irreducible if and only if one of the following holds:

(a) $M = U_3(4) : 2$, $\varphi$ is the unique irreducible character of degree 64 when $\ell = 3$ or $\varphi$ is the unique irreducible character of degree 65 when $\ell \neq 2, 3$;
(b) $M = J_2$, $\varphi$ is any of the two irreducible characters of degree 300 when $\ell \neq 2, 3, 7$.

Lemma 2.4. When $G = 2 \cdot G_2(4)$, the universal cover of $G_2(4)$, $\varphi|_M$ is irreducible if and only if one of the following holds:

(a) $M = 2.P$ or $2.Q$ where $P$, $Q$ are maximal parabolic subgroups of $G_2(4)$, $\varphi$ is the unique irreducible character of degree 12;
(b) $M = 2.(U_3(4) : 2)$, $\varphi$ is the unique irreducible character of degree 12 or $\varphi$ is any of the two irreducible characters of degree 104 when $\ell \neq 2, 5$;
(c) $M = 2.(SL_3(4) : 2)$, $\varphi$ is the unique irreducible character of degree 12 when $\ell \neq 2, 3$.

In the case of complex representations, Theorem 1.1 was proved by Saxl [S]. Similar results for the Suzuki group $^2B_2(q)$ and the Ree group $^2G_2(q)$ are obtained in Section 6.

The paper is divided into six sections. In Section 2, we set up some notation and state some lemmata which will be used later. In Section 3, we collect the degrees of irreducible Brauer characters and give the formulas for the first and second degrees of $G$. Sections 4 and 5 are devoted to prove the main results. In the last section, we state and prove the results for the Suzuki group $^2B_2(q)$ and the Ree group $^2G_2(q)$.

2. Preliminaries

Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell$. Given a finite group $X$, we denote by $d_\ell(X)$, $d_{2,\ell}(X)$, and $m_\ell(X)$ the smallest, the second smallest, and the largest degrees, respectively, of irreducible $\mathbb{F}X$-representations of degree larger than 1. When $\ell = 0$, we use the notation $d_C(X)$, $d_{2,C}(X)$, and $m_C(X)$ instead of $d_0(X)$, $d_{2,0}(X)$, and $m_0(X)$. If $\chi$ is a complex character of $X$, we denote by $\overline{\chi}$ the restriction of $\chi$ to $\ell$-regular elements of $X$. Throughout this paper, Irr$(X)$ is the set of irreducible complex characters of $X$; Irr$_\ell(X)$ is the set of irreducible $\ell$-Brauer characters of $X$; and $Z_n$ is the cyclic group of order $n$.

The following lemmata are well known and we omit most of their proofs.

Lemma 2.1. Let $G$ be a finite group and $H$ be a subgroup of $G$. Let $\Phi$ be an irreducible $\mathbb{F}G$-representation of degree larger than 1 such that $\Phi|_H$ is also irreducible. Then $m_\ell(H) \geq \deg(\Phi) \geq d_\ell(G)$.

Lemma 2.2. Let $G$ be a finite group. Then $m_\ell(G) \leq m_C(G)$. Furthermore, $m_C(G) \leq \sqrt{|G/Z(G)|}$.

Lemma 2.3. Let $G$ be a simple group and $V$ an irreducible $\mathbb{F}G$-module such that $\dim(V) > 1$. Then $Z_G(V) := \{g \in G \mid g|_V = \lambda \cdot \text{Id}_V \text{ for some } \lambda \in \mathbb{F}\} = 1$.

Lemma 2.4. Let $G$ be a finite group and $1 \neq A \trianglelefteq H \trianglelefteq G$. Let $V$ be a faithful irreducible $\mathbb{F}G$-module.

(i) Suppose that $C_V(A) := \{v \in V \mid a(v) = v \text{ for every } a \in A\} \neq 0$. Then $V|_H$ is reducible.
(ii) Suppose that $O_\ell(H) \neq 1$. Then $V|_H$ is reducible.
Proof. (i) Assume the contrary that $V|_H$ is irreducible. By Clifford's theorem, $V|_A = e \bigoplus_{i=1}^t V_i$ where $e$ is the multiplicity of $V_i$ in $V$ and $\{V_1, \ldots, V_t\}$ is the orbit of $V_i$ under the action of $H$ on the set of all irreducible $FA$-modules. Since $A$ acts trivially on $C_V(A) \subseteq V$, one of the $V_i$s is the trivial module. Therefore all $V_1, \ldots, V_t$ are trivial $A$-modules and $A$ acts trivially on $V$. This contradicts the faithfulness of $V$.

(ii) Assume that $V|_H$ is irreducible. Then $O_\ell(H)$ acts trivially on $V$. Hence $C_V(O_\ell(H)) = V$ and we get a contradiction by part (i). \qed

Lemma 2.5. Let $\tau$ be an automorphism of a finite group $G$ which fuses two conjugacy classes of subgroups of $G$ with representatives $H_1$, $H_2$. Let $A$ and $B$ be subsets of $\text{IBr}_\ell(G)$ such that $\tau(A) = B$, $\tau(B) = A$.

(i) Assume $\varphi|_{H_1}$ is irreducible (resp. reducible) for all $\varphi \in A \cup B$. Then $\varphi|_{H_2}$ is irreducible (resp. reducible) for all $\varphi \in A \cup B$.

(ii) Assume $\varphi|_{H_1}$ is irreducible for all $\varphi \in A$ and $\varphi|_{H_1}$ is reducible for all $\varphi \in B$. Then $\varphi|_{H_2}$ is reducible for all $\varphi \in A$ and $\varphi|_{H_2}$ is irreducible for all $\varphi \in B$.

Proof. It is enough to show that $\varphi|_{H_1}$ is irreducible (resp. reducible) for all $\varphi \in A$ if and only if $\varphi|_{H_2}$ is irreducible (resp. reducible) for all $\varphi \in B$. This is true because for every $\varphi \in A$, we have $\varphi|_{H_1} = \psi \circ \tau|_{O_\ell(H)} = (\psi|_{H_2}) \circ \tau$ for some $\psi \in B$. \qed

Lemma 2.6. Let $G$ be a finite group and $\chi \in \text{Irr}(G)$. Let $H$ be a normal $\ell'$-subgroup of $G$ and suppose that $\chi|_H = \sum_{i=1}^t \theta_i$, where $\theta_1$ is irreducible and $\theta_1, \theta_2, \ldots, \theta_t$ are the distinct $G$-conjugates of $\theta_1$. Then $\hat{\chi}$ is also irreducible.

Proof. We have $\hat{\chi}|_H = \sum_{i=1}^t \theta_i$ as $H$ is an $\ell'$-group. Let $\psi$ be an irreducible constituent of $\hat{\chi}$. Then there is some $i \in \{1, 2, \ldots, t\}$ such that $\theta_i$ is an irreducible constituent of $\psi|_H$. Therefore, by Clifford’s theorem, all $\theta_1, \theta_2, \ldots, \theta_t$ are contained in $\psi|_H$. This implies that $\hat{\chi}|_H = \psi|_H$. So $\hat{\chi} = \psi$ and $\hat{\chi}$ is irreducible. \qed

Lemma 2.7. Let $G$ be a finite group. Suppose that the universal cover of $G$ is $M.G$ where $M$ is the Schur multiplier of $G$. Then every maximal subgroup of $M.G$ is the pre-image of a maximal subgroup of $G$ under the natural projection $\pi : M.G \to G$.

Lemma 2.8. (See [F].) Let $G$ be a finite group and $H \triangleleft G$. Suppose $|G: H| = p$ is prime and $\chi \in \text{IBr}_\ell(G)$. Then either

(i) $\chi|_H$ is irreducible or

(ii) $\chi|_H = \sum_{i=1}^p \theta_i$, where the $\theta_i$s are distinct and irreducible.

Lemma 2.9. (See [OT].) Let $B$ be an $\ell$-block of group $G$. Assume that all $\chi \in B \cap \text{Irr}(G)$ are of same degree. Then $B \cap \text{IBr}_\ell(G) = \{\phi\}$ and $\hat{\chi} = \phi$ for every $\chi \in B \cap \text{Irr}(G)$.

Lemma 2.10. (See [T2, Theorem 1.6].) Let $G$ be a finite group of Lie type, of simply connected type. Assume that $G$ is not of type $A_1$, $2A_2$, $2B_2$, $2G_2$, and $B_2$. If $Z$ is a long-root subgroup and $V$ is a nontrivial irreducible $G$-module, then $Z$ must have nonzero fixed points on $V$. 

Table 1
The degrees of irreducible complex characters of \( G_2(q) \)

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>( q^6 )</td>
</tr>
<tr>
<td>13, 14</td>
<td>( \frac{1}{2}q(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>15</td>
<td>( \frac{1}{2}q(q + 1)^2(q^2 - q + 1) )</td>
</tr>
<tr>
<td>16</td>
<td>( \frac{1}{6}q(q + 1)^2(q^2 + q + 1) )</td>
</tr>
<tr>
<td>17</td>
<td>( \frac{1}{6}q(q - 1)^2(q^2 + q + 1) )</td>
</tr>
<tr>
<td>18</td>
<td>( \frac{1}{6}q(q - 1)^2(q^2 - q + 1) )</td>
</tr>
<tr>
<td>19, 19</td>
<td>( \frac{1}{2}q(q - 1)^2(q + 1)^2 )</td>
</tr>
<tr>
<td>31</td>
<td>( q^3(q^3 + \epsilon) )</td>
</tr>
<tr>
<td>32</td>
<td>( q^3 + \epsilon )</td>
</tr>
<tr>
<td>33</td>
<td>( q(q + \epsilon)(q^3 + \epsilon) )</td>
</tr>
<tr>
<td>21</td>
<td>( q^2(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>22</td>
<td>( q^4 + q^2 + 1 )</td>
</tr>
<tr>
<td>23, 24</td>
<td>( q(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>1a, 1b, 2a, 2b</td>
<td>( q(q \pm 1)(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>1a', 1b', 2a', 2b'</td>
<td>( (q \pm 1)(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>1, 2</td>
<td>( (q \pm 1)^2(q^4 + q^2 + 1) )</td>
</tr>
<tr>
<td>2a, 2b</td>
<td>( q^6 - 1 )</td>
</tr>
</tbody>
</table>
| 3, 6      | \( (q^2 - 1)^2(q^2 \mp q + 1) \)

where:

(i) \( \epsilon \equiv q \) (mod 3),
(ii) \( X_{21}, X_{22}, X_{23}, X_{24} \) appear only if \( q \) is odd,
(iii) \( X_{31}, X_{32}, X_{33} \) appear only if \( q \) is not divisible by 3.

3. The degrees of irreducible Brauer characters of \( G_2(q) \)

In this section, we will recall the value of \( d_\ell(G) \) and determine the value of \( d_{2,\ell}(G) \) when \( \ell \nmid q \). The degrees of irreducible complex characters of \( G_2(q) \) can be read off from [CR,E,EY] and are listed in Table 1.

From the table, we get that if \( q \geq 5 \) then

\[
d_\mathbb{C}(G) = \begin{cases} 
q^3 + 1, & q \equiv 1 \text{ (mod 3)}, \\
q^3 - 1, & q \equiv 2 \text{ (mod 3)}, \\
q^4 + q^2 + 1, & q \equiv 0 \text{ (mod 3)},
\end{cases} \tag{3.1}
\]

and

\[
d_{2,\mathbb{C}}(G) = \begin{cases} 
\frac{1}{6}q(q - 1)^2(q^2 - q + 1), & p = 2, 3 \text{ or } q = 5, 7, \\
q^4 + q^2 + 1, & p \geq 5, q > 7.
\end{cases} \tag{3.2}
\]

Moreover, \( \text{Irr}(G) \) contains a unique character of degree larger than 1 but less than \( d_{2,\mathbb{C}}(G) \) and this character has degree \( d_\mathbb{C}(G) \).
we refer to $[H1,S2,S3]$. In these tables, from $[H1,HS1,HS2,S2]$ and $[S3]$. They are listed in Tables 2, 3 for $q ≢ 1 \pmod{3}$. Therefore, if $3 \mid q$ and $0 ≤ α ≤ 2q$ if $3 \mid q$.

(ii) $0 ≤ β ≤ \frac{1}{2}(q + 2)$.

(iii) $1 ≤ γ ≤ \frac{1}{2}(q + 1)$.

Therefore, if $3 \mid q$ then $\varphi_{12}(1) ≥ \frac{1}{2}(q - 1)^2(q + 1)(q^3 + 2q^2 + q + 3)$ and if $3 \mid q$ then $\varphi_{12}(1) ≥ \frac{1}{2}(q - 1)^2(q^5 + 2q^2 + 4q + 3)$. Moreover, when $4 \mid (q + 1)$ and $q ≡ -1 \pmod{3}$, we have $\varphi_{31}(1) ≥ \frac{1}{2}(q - 1)^2(q^2 + q + 1)(2q^2 + 2q + 3)$ (cf. [HS2]).

Table 3
The degrees of irreducible 3-Brauer characters of $G_2(q)$ with $3 \mid q$

<table>
<thead>
<tr>
<th>Character</th>
<th>$3 \mid (q - 1), 0 ≤ α ≤ 1$</th>
<th>$3 \mid (q + 1), 1 ≤ α ≤ q + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_{12}$</td>
<td>$\frac{1}{6}(q - 1)^2(6q^4 + (9 - \beta - 2\gamma)q^3$</td>
<td>$\frac{1}{6}(q - 1)^2(6q^4 + (11 - \alpha - 2\beta - 3\gamma)q^3 + (9 + \beta - 4\gamma)q^2 + (9 - \beta - 2\gamma)q + 6)$</td>
</tr>
<tr>
<td>$\varphi_{14}$</td>
<td>$\frac{1}{6}q(q^2 - q + 1)(1 - \alpha)q^2 + (4 + 2\alpha)q + (1 - \alpha)$</td>
<td>$\frac{1}{6}(q^2 - 1)(q^3 + 3q^2 - q + 6)$</td>
</tr>
<tr>
<td>$\varphi_{15}$</td>
<td>$\frac{1}{2}q^3 + q^4 + q^2 + q - 2$</td>
<td>$\frac{1}{2}q(q + 1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\varphi_{21}$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
<td>$q^2(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{22}$</td>
<td>$q^4 + q^2 + 1$</td>
<td>$q^4 + q^2 + 1$</td>
</tr>
<tr>
<td>$\varphi_{23}$, $\varphi_{24}$</td>
<td>$q(q^4 + q^2 + 1)$</td>
<td>$q(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{21}$, $\varphi_{22}$</td>
<td>$q(q + 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)(q^4 + q^2 + 1)$</td>
</tr>
<tr>
<td>$\varphi_{23}$, $\varphi_{24}$</td>
<td>$q(q - 1)(q^4 + q^2 + 1)$</td>
<td>$(q - 1)^2(q^4 + q^2 + 1)$</td>
</tr>
</tbody>
</table>

where:

(i) when $3 \mid (q - 1)$, $\varphi_{14}(1) ∈ \{\frac{1}{6}q(q^2 - q + 1)(q^2 + 4q + 1), q^2(q^2 - q + 1)\}$ and $\varphi_{12} ≥ \frac{1}{2}(q - 1)^2(q^4 + 2q^3 + 3q + 2)$;

(ii) when $3 \mid (q + 1)$, $\varphi_{12} ≥ \frac{1}{2}(q - 1)^2(q^2 + q + 1)$ (cf. [HS1]).

The degrees of irreducible $\ell$-Brauer characters of $G_2(q)$ when $\ell \mid |G|$ and $\ell \not| q$ can be read off from [H1,HS1,HS2,2], and [S3]. They are listed in Tables 2, 3 for $\ell = 2$ and $\ell = 3$. When $\ell ≥ 5$, we refer to [H1,S2,S3]. In these tables, $q ≡ ε \pmod{3}$. Moreover, $\varphi_{21}$, $\varphi_{22}$, $\varphi_{23}$, $\varphi_{24}$ appear only if $q$ is odd and $\varphi_{31}$, $\varphi_{32}$, $\varphi_{33}$ appear only if $q$ is not divisible by 3.
Note that, besides the characters listed in Tables 2, 3, the degrees of the remaining irreducible Brauer characters of $G_2(q)$: $\varphi_{11}, \varphi_{17}, \varphi_{18}, \varphi_{19}, \varphi_{19}', \varphi_{19}, \varphi_{1a}, \varphi_{1b}, \varphi_2, \varphi_2, \varphi_3, \varphi_3, \varphi_5$ are the same as the degrees of complex characters with the same indices.

By comparing the degrees of characters directly, we easily get the value of $\delta_\ell(G)$. We include the case when $\ell \nmid |G|$ in the following formula, where $q \geq 5$ and $\ell \nmid q$.

$$\delta_\ell(G) = \begin{cases} q^3 + 1, & q \equiv 1 \pmod{3}, \ell \neq 3, \\ q^3, & q \equiv 1 \pmod{3}, \ell = 3, \\ q^3 - 1, & q \equiv 2 \pmod{3}, \forall \ell, \\ q^4 + q^2, & q \equiv 0 \pmod{3}, \ell = 2, \\ q^4 + q^2 + 1, & q \equiv 0 \pmod{3}, \ell \neq 2. \end{cases}$$  \hspace{1cm} (3.3)

When $\ell \nmid |G|$, we know that $\text{IBr}_\ell(G) = \text{Irr}(G)$ and the value of $\delta_{2,C}(G)$ is given in formula (3.2). Formula (3.2) and direct comparison of degrees of $\ell$-Brauer characters when $\ell \mid |G|$ yields the following theorem. We omit the details of this direct computation.

**Theorem 3.1.** Let $G = G_2(q)$ with $q \geq 5$. We have

$$\delta_{2,2}(G) = \begin{cases} \frac{1}{6}q(q-1)^2(q^2 - q + 1), & p = 3 \text{ or } q = 5, 7, \\ q^4 + q^2, & p \geq 5, q \geq 11, \end{cases}$$

$$\delta_{2,3}(G) \begin{cases} = \frac{1}{6}q(q-1)^2(q^2 - q + 1), & q = 5, 7, \text{ or } p = 2, q \equiv -1 \pmod{3}, \\ = q^4 + q^2 + 1, & p \geq 5, q \equiv -1 \pmod{3} \text{ and } q \geq 11, \\ \geq q^4 - q^3 + q^2, & q \geq 13 \text{ and } q \equiv 1 \pmod{3}, \end{cases}$$

and if $\ell = 0$ or $\ell \equiv 5$, $\ell \nmid q$ then

$$\delta_{2,\ell}(G) = \delta_{2,C}(G) = \begin{cases} \frac{1}{6}q(q-1)^2(q^2 - q + 1), & p = 2, 3 \text{ or } q = 5, 7, \\ q^4 + q^2 + 1, & p \geq 5, q \geq 11. \end{cases}$$

Moreover, $\text{IBr}_\ell(G)$ contains a unique character denoted by $\psi$ of degree larger than 1 but less than $\delta_\ell(G)$ and this character has degree $\delta_\ell(G)$.

From the results about blocks and Brauer trees of $G_2(q)$ in [H1,HS1,HS2,S1,S2], we see that if $3 \nmid q$ then $\psi = \overline{X}_{32}^2$ in all cases except when $\ell = 3$ and $q \equiv 1 \pmod{3}$ where $\psi = \overline{X}_{32}^2 - 1_G$. If $3 \mid q$ then $\psi = \overline{X}_{22}^2$ except when $\ell = 2$ where $\psi = \overline{X}_{22}^2 - 1_G$. We also notice that $\varphi_{18} = \overline{X}_{18}^2$ is irreducible in all cases.

### 4. Proof of main results

In next lemmata, we use the notation in [CR,GE,G,H2,SE].

**Lemma 4.1.**

(i) When $q \equiv -1 \pmod{3}$, the character $X_{32}|_{SL_3(q)}$ is irreducible and equal to $\chi_{q^3-1}^{(q^2-1)}$.

(ii) When $q \equiv 1 \pmod{3}$, the character $X_{32}|_{SU_3(q)}$ is irreducible and equal to $\chi_{q^3+1}^{(q^2-1)}$. 

Proof. (i) First let us consider the case when \( q \) is odd and \( q \equiv -1 \pmod{3} \). We need to find the fusion of conjugacy classes of \( SL_3(q) \) in \( G \). By [SF, p. 487], \( SL_3(q) \) has the following conjugacy classes: \( C_1^{(0)}, C_2^{(0)}, C_3^{(0,0)}, C_4^{(k)}, C_5^{(k)}, C_6^{(k,l,m)}, C_7^{(k)}, \) and \( C_8^{(k)} \). We denote by \( K(x) \) the conjugacy class containing an element \( x \in G \). For the representatives of the conjugacy classes of \( G \), we refer to [CR, pp. 396–398]. Then we have \( C_1^{(0)} \subseteq K(1), C_2^{(0)} \subseteq K(u_1), C_3^{(0,0)} \subseteq K(u_0), C_4^{(k)} \subseteq K(k_2) \cup K(h_1b), C_5^{(k)} \subseteq K(k_2,1) \cup K(h_{1b,1}), C_6^{(k,l,m)} \subseteq K(h_1), C_7^{(k)} \subseteq K(h_b) \cup K(h_{2b}) \) and \( C_8^{(k)} \subseteq K(h_3) \). Taking the values of \( X_{32} \) and \( \chi_{q^3-1}^{(a^2-1)} \) on these classes, we get

\[
\begin{align*}
\chi_{q^3-1}^{(a^2-1)}(C_1^{(0)}) &= X_{32}(1) = q^3 - 1, \\
\chi_{q^3-1}^{(a^2-1)}(C_2^{(0)}) &= X_{32}(u_1) = -1, \\
\chi_{q^3-1}^{(a^2-1)}(C_3^{(0,0)}) &= X_{32}(u_0) = -1, \\
\chi_{q^3-1}^{(a^2-1)}(C_4^{(k)}) &= X_{32}(k_2) = X_{32}(h_{1b}) = q - 1, \\
\chi_{q^3-1}^{(a^2-1)}(C_5^{(k)}) &= X_{32}(k_{2,1}) = X_{32}(h_{1b,1}) = -1, \\
\chi_{q^3-1}^{(a^2-1)}(C_6^{(k,l,m)}) &= X_{32}(h_1) = 0, \\
\chi_{q^3-1}^{(a^2-1)}(C_7^{(k)}) &= X_{32}(h_b) = X_{32}(h_{2b}) = -\left(\omega^k + w^{-k}\right), \\
\chi_{q^3-1}^{(a^2-1)}(C_8^{(k)}) &= X_{32}(h_3) = 0, 
\end{align*}
\]

where \( \omega \) is a primitive cubic root of unity. Therefore, \( X_{32}|_{SL_3(q)} = \chi_{q^3-1}^{(a^2-1)} \). The case \( q \) even is proved similarly by using [EY].

(ii) This part is proved similarly by using [CR, pp. 396–398] and [G, p. 565].

Lemma 4.2. Suppose that \( q \equiv -1 \pmod{3} \). Then \( \chi_{q^3-1}^{(a^2-1)} \in \text{IBr}_\ell(SL_3(q)) \) for \( \ell \nmid q \).

Proof. Since \( q \equiv -1 \pmod{3} \), \( GL_3(q) = SL_3(q) \times Z(GL_3(q)) \) with \( Z(GL_3(q)) \simeq \mathbb{Z}_{q-1} \). Let \( V \) be a \( CGL_3(q) \)-module affording the irreducible character \( \chi_{q^3-1}^{(a^2-1)} \cdot 1_{\mathbb{Z}_{q-1}} \in \text{Irr}(GL_3(q)) \). We have \( \dim V = q^3 - 1 \). According to [GT, Section 4], \( V \) has the form \( S_\mathbb{C}(s, (1)) \circ S_\mathbb{C}(t, (1)) \uparrow G \) where \( s \in \mathbb{P} \mathbb{F}_q^2 \) and \( t \) has degree 2 over \( \mathbb{F}_q \). Using Corollary 2.7 of [GT], since 1 and 2 are co-prime, we see that \( V \) is irreducible in any cross characteristic. This implies that \( V|_{SL_3(q)} \) is also irreducible in any cross characteristic and therefore \( \chi_{q^3-1}^{(a^2-1)} \in \text{IBr}_\ell(SL_3(q)) \) for every \( \ell \nmid q \).

Note. 2-Brauer characters of \( SU_3(q) \) were not considered in [G].
Lemma 4.3. Suppose that \( q \equiv 1 \pmod{3} \) and \( q \) is odd. Then 
\[ \chi_{q^3+1}(\frac{q^2-1}{q^3+1}) \in \text{IBr}_2(SU_3(q)). \]

**Proof.** We denote the character \( \chi_{q^3+1}(\frac{q^2-1}{q^3+1}) \) by \( \rho \) for short. Since \( q \equiv 1 \pmod{3} \), \( GU_3(q) = SU_3(q) \times Z(GU_3(q)) \) with \( Z(GU_3(q)) \cong \mathbb{Z}_{q+1} \). Hence, \( SU_3(q) \) has the same degrees of irreducible Brauer characters as \( GU_3(q) \) does. It is shown in [TZ] that every \( \varphi \in \text{IBr}_2(GU_3(q)) \) either lifts to characteristic 0 or \( \varphi(1) = q(q^2 - q + 1) - 1 \). This and the character table of \( SU_3(q) \) (cf. [G]) gives the possible values for degrees of 2-Brauer characters of \( SU_3(q) \): 1, \( q^2 - q \), \( q^2 - q + 1 \), \( q(q^2 - q + 1) - 1 \), \( q(q^2 - q + 1) \), \( q(q^2 - q + 1) \), \( q^3 \), \( q^3 + 1 \), and \( (q + 1)^2(q - 1) \).

Assume \( \rho \) is reducible. Then \( \rho \) is the sum of more than one irreducible characters of \( SU_3(q) \). Hence, \( \rho(1) = q^3 + 1 \) is the sum of more than one of the values listed above. It follows that \( \rho \) must include a character of degree either 1, or \( q^2 - q \), or \( q^2 - q + 1 \). Once again, this character lifts to a complex character that we denote by \( \alpha \). Clearly, \( \rho \) and \( \alpha \) belong to the same 2-block of \( SU_3(q) \). We will use central characters to show that this cannot happen.

Let \( R \) be the full ring of algebraic integers in \( \mathbb{C} \) and \( \pi \) a maximal ideal of \( R \) containing \( 2R \). It is known that \( \alpha \) and \( \rho \) are in the same 2-block if and only if

\[ \omega_{\rho}(K) - \omega_{\alpha}(K) \in \pi \tag{4.4} \]

where \( K \) is any class sum and \( \omega_{\chi} \) is the central character associated with \( \chi \). The value of \( \omega_{\chi} \) on a class sum is

\[ \omega_{\chi}(K) = \frac{\chi(g)|K|}{|\chi(1)} \]

where \( |K| \) is the conjugacy class with class sum \( K \) and \( g \) is an element in \( K \). Therefore, (4.4) implies that

\[ \frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| \in \pi \tag{4.5} \]

where \( |g^G| \) denotes the length of the conjugacy class containing \( g \in G \).

First assume that \( \alpha \) is the trivial character. In (4.5), take \( g \) to be any element in the conjugacy class \( C_1^{(1)} \) (cf. [G]), we have

\[ \frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| = \frac{1}{q^3+1}q^3(q^3+1) - \frac{1}{q^3+1}q^3(q^3+1) = -q^3 - q^3 = -q^3 \in \pi. \]

Since \( -q^3 \) is an odd number, we again get a contradiction.

Secondly, if \( \alpha(1) = q^2 - q \) then \( \alpha = \chi_{q^2-q} \) as denoted in [G]. In (4.5), take \( g \) to be any element in the conjugacy class \( C_1^{(1)} \), we have

\[ \frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| = \frac{1}{q^3+1}q^3(q^3+1) - \frac{0}{q^2-q}q^3(q^3+1) = -q^3 \in \pi. \]

Since \( -q^3 \) is an odd number, we again get a contradiction.

Finally, if \( \alpha(1) = q^2 - q + 1 \) then \( \alpha = \chi_{q^2-q+1}^{(u)} \) for some \( 1 \leq u \leq q \). In (4.5), take \( g \) to be any element in the conjugacy class \( C_{q+1}^{(q+1)} \). Then we have

\[ \frac{\rho(g)}{\rho(1)}|g^G| - \frac{\alpha(g)}{\alpha(1)}|g^G| = \frac{1}{q^3+1}q^3(q^3+1) - \frac{1}{q^2-q+1}q^3(q^3+1) = -q^3 - q^3 = -q^3 \in \pi. \]

This is an odd number, a contradiction. \( \square \)

**Theorem 4.4** (Reduction Theorem). Let \( G = G_2(q), q = p^n, q \geq 5, \) and \( p \) a prime number. Let \( \Phi \) be an absolutely irreducible representation of \( G \) in cross characteristic \( \ell \) and \( M \) a maximal
subgroup of $G$. Assume that $\deg(\Phi) > 1$ and $\Phi|_M$ is also absolutely irreducible. Then $M$ is $G$-conjugate to one of the following groups:

(i) maximal parabolic subgroups $P_a, P_b$,
(ii) $SL_3(q) : 2, SU_3(q) : 2$,
(iii) $G_2(q_0)$ with $q = q_0^2$, $p \neq 3$.

**Proof.** By Lemmata 2.1 and 2.2, we have $d_\ell(G) \leq \deg(\Phi) \leq m_\ell(M) \leq m_C(M) \leq \sqrt{|M|}$. Hence, $d_\ell(G) \leq \sqrt{|M|}$. Moreover, from formulas (3.1) and (3.3), we have $d_\ell(G) \geq q^3 - 1$ if $3 \mid q$ and $d_\ell(G) \geq q^4 + q^2$ if $3 \mid q$ for every $\ell \mid q$. Therefore,

$$\sqrt{|M|} \geq \begin{cases} q^3 - 1 & \text{if } 3 \mid q, \\ q^4 + q^2 & \text{if } 3 \mid q. \end{cases} \quad (4.6)$$

Here, we will only give the proof for the case $p \geq 5$. The proofs for $p = 2$ and $p = 3$ are similar. According to [K1], if $M$ is a maximal subgroup of $G$ with $p \geq 5$ then $M$ is $G$-conjugate to one of the following groups:

(1) $P_a, P_b$, maximal parabolic subgroups,
(2) $(SL_2(q) \circ SL_2(q)) \cdot 2$, involution centralizer,
(3) $2^3 \cdot L_3(2)$, only when $p = q$,
(4) $SL_3(q) : 2, SU_3(q) : 2$,
(5) $G_2(q_0), q = q_0^\alpha$, $\alpha$ prime,
(6) $PGL_2(q)$, $p \geq 7, q \geq 11$,
(7) $L_2(8), p \geq 5$,
(8) $L_2(13), p \neq 13$,
(9) $G_2(2), q = p \geq 5$,
(10) $J_1, q = 11$.

Consider for instance the case (5) with $\alpha \geq 3$. Then $\sqrt{|M|} = \sqrt{q_0^6(q_0^6 - 1)(q_0^2 - 1)} < q_0^7 \leq q^{7/3} < q^3 - 1$ for every $q \geq 5$. This contradicts (4.6). The cases (2), (3), (6)–(10) are excluded similarly. □

**Proof of Theorem 1.1.** By the Reduction Theorem, $M$ is $G$-conjugate to one of the following subgroups:

(i) maximal parabolic subgroups $P_a, P_b$,
(ii) $SL_3(q) : 2, SU_3(q) : 2$,
(iii) $G_2(q_0)$ with $q = q_0^2, p \neq 3$.

Now we will proceed case by case.

**Case 1.** $M = P_a$.

Let $Z := Z(P_a')$, the center of the derived subgroup of $P_a$. We know that $P_a$ is the normalizer of $Z$ in $G$ and therefore $Z$ is nontrivial. In fact, $Z$ is a long-root subgroup of $G$. Let $V$ be an
irreducible $G$-module affording the character $\varphi$. By Lemma 2.10, $Z$ must have nonzero fixed points on $V$. In other words, $C_V(Z) = \{v \in V \mid a(v) = v \text{ for every } a \in Z\} \neq 0$. Therefore $V|_{P_b}$ is reducible by Lemma 2.4(i).

**Case 2. $M = P_b$.**

Using the results about character tables of $P_b$ in [AH,E], and [EY], we have $m_C(P_b) = q(q-1)(q^2-1)$ for $q \geq 5$. Therefore, if $\varphi|_{P_b}$ is irreducible then $\varphi(1) \leq q(q-1)(q^2-1)$. If $3 \mid q$ then $d(G) \geq q^4 + q^2$ because of formula (3.3). Then we have $d(G) \geq q^4 + q^2 > q(q-1)(q^2-1)$ and this cannot happen. So $q$ must be congruent to $3$.

It is easy to check that $m_C(P_b) < d_{2,\ell}(G)$ for every $q \geq 8$. Therefore, when $q \geq 8$, the inequality $\varphi(1) \leq q(q-1)(q^2-1)$ can hold only if $\varphi$ is the nontrivial character of smallest degree. When $q = 5$ or $7$, checking directly, we see that besides the nontrivial character of smallest degree, $\varphi$ can be $\varphi_{18} = X_{18}$ of degree $\frac{1}{6}q(q-1)^2(q^2-1)$. Recall that $|P_b| = q^6(q^2-1)(q-1)$, which is not divisible by $X_{18}(1)$. Therefore $X_{18}|_{P_b}$ is reducible and so is $\varphi_{18}|_{P_b}$ for $q = 5$ or $7$.

We have shown that the unique possibility for $\varphi$ is the character $\psi$ of smallest degree when $3 \mid q$.

Now note that $X_{32}(1) = (q^3 + 1)\mid |P_b|$ and therefore $X_{32}|_{P_b}$ must be reducible. If $\ell = 3$ and $q \equiv 1 \pmod{3}$ then $\psi = X_{32} - 1_G$. Assume that $\psi|_{P_b} = X_{32} - 1_G$ is irreducible. The reducibility of $X_{32}|_{P_b}$ and the irreducibility of $X_{32}|_{P_b} - 1_G$ imply that $X_{32}|_{P_b} = \lambda + \mu$ where $\lambda = 1_G$, $\mu \in \text{Irr}(P_b)$ and $\mu \in \text{IBr}_3(P_b)$. We then have $\mu(1) = X_{32}(1) - 1 = q^3$. Inspecting the character tables of $P_b$ given in [AH] and [EY], we see that there is no reducible complex character of $P_b$ of degree $q^3$ and we get a contradiction. If $\ell \neq 3$ or $q$ is not congruent to $1$ modulo $3$ then $\psi = X_{32}$. Therefore $\psi|_{P_b} = X_{32}|_{P_b}$, which is reducible as noted above.

**Case 3. $M = SL_3(q) : 2$.**

From [SF], we know that $m_C(SL_3(q)) = (q+1)(q^2 + q + 1)$ for every $q \geq 5$. Therefore, $m_C(SL_3(q) : 2) \leq 2(q+1)(q^2 + q + 1)$. Since $\varphi|_{SL_3(q):2}$ is irreducible, $\varphi(1) \leq 2(q+1)(q^2 + q + 1)$. Similarly as the previous case, we have $q^4 + q^2 > 2(q+1)(q^2 + q + 1)$ for every $q \geq 5$ and so $q$ is not divisible by $3$.

By Theorem 3.1, the inequality $\varphi(1) \leq 2(q+1)(q^2 + q + 1)$ can hold only if $\varphi$ is the character $\psi$ of smallest degree or $\varphi_{18} = X_{18}$ when $q = 5$. We have $X_{18}(1) = 280$, $|SL_3(5) : 2| = 744,000$ and therefore $X_{18}|_{SL_3(5)} : 2| = 5$. Hence, $X_{18}|_{SL_3(q):2}$ is irreducible and so is $\varphi_{18}|_{SL_3(q):2}$ when $q = 5$. Again, the unique possibility for $\varphi$ is $\psi$ when $3 \mid q$.

If $\ell = 3$ and $q \equiv 1 \pmod{3}$ then $\psi = X_{32} - 1_G$. Assume that $\psi|_{SL_3(q):2}$ is irreducible. Let $V$ be an irreducible $G$-module in characteristic $3$ affording the character $\psi$. Then $V|_{SL_3(q):2}$ is an irreducible $(SL_3(q) : 2)$-module. Let $\sigma$ be a generator for the multiplicative group $\mathbb{F}_q^\times$ and $I$ be the identity matrix in $SL(3,\mathbb{F}_q)$. Consider the matrix $T = \sigma^2 \cdot I$. We have $\langle T \rangle = Z(SL_3(q))$ and hence $\langle T \rangle \leq SL_3(q) : 2$. Since $\text{ord}(T) = 3$ and $\langle T \rangle \leq O_3(SL_3(q) : 2)$, it follows that $O_3(SL_3(q) : 2)$ is nontrivial. By Lemma 2.4(ii), $V|_{SL_3(q):2}$ is reducible and we get a contradiction. If $\ell \neq 3$ and $q \equiv 1 \pmod{3}$ then $\psi = X_{32}$. Note that $X_{32}(1) = q^3 + 1$ and $|SL_3(q) : 2| = 2q^3(q^2 - 1)(q^2 - 1)$. Therefore $X_{32}(1) \mid |SL_3(q) : 2|$ for every $q \geq 5$. It follows that $X_{32}|_{SL_3(q):2}$ as well as $\psi|_{SL_3(q):2}$ are reducible.
It remains to consider $q \equiv -1 \pmod{3}$ and therefore $\psi = \hat{X}_{32}$. By Lemmata 4.1 and 4.2, we get that $\psi|_{SL_3(q)} = X_{32}|_{SL_3(q)} = \chi_{q^3 - 1}^{(q^2 - 1)} \in IBr_\ell(SL_3(q))$ for $\ell \nmid q$, as we claim in the item (i) of Theorem 1.1.

**Case 4.** $M = SU_3(q) : 2$.

According to [SF], $m_\mathbb{C}(SU_3(q)) = (q + 1)^2(q - 1)$ and therefore $m_\mathbb{C}(SU_3(q) : 2) \leq 2(q + 1)^2(q - 1)$ for every $q \geq 5$. Hence $\varphi(1) \leq 2(q + 1)^2(q - 1)$ by the irreducibility of $\varphi|_{SU_3(q) : 2}$. Again, $q$ must be coprime to 3.

By Theorem 3.1, the inequality $\varphi(1) \leq 2(q + 1)^2(q - 1)$ can hold only if $\varphi$ is the character $\psi$ of smallest degree or $\varphi_{18}$ of degree $\frac{1}{6}q(q - 1)^2(q^2 - q + 1)$ when $q = 5$. By [G], the degrees of irreducible characters of $SU_3(5)$ are: 1, 20, 125, 21, 105, 84, 126, 144, 28 and 48. When $q = 5$, $X_{18}(1) = 280$. Therefore, $X_{18}|_{SU_3(5)}$ is the sum of at least 3 irreducible characters. Since $SU_3(5)$ is a normal subgroup of index 2 of $SU_3(5) : 2$, by Clifford’s theorem, $X_{18}|_{SU_3(5) : 2}$ is reducible when $q = 5$. This implies that $\varphi_{18}$ is also reducible when $q = 5$. So, the unique possibility for $\varphi$ is $\psi$ when $3 \nmid q$.

If $q \equiv -1 \pmod{3}$ then $\psi = \hat{X}_{32}$. Note that $|SU_3(q) : 2| = 2q^3(q^3 + 1)(q^2 - 1)$, which is not divisible by $q^3 - 1$ for every $q \geq 5$. Hence $X_{32}|_{SU_3(q) : 2}$ is reducible and so is $\psi|_{SU_3(q) : 2}$. It remains to consider $q \equiv 1 \pmod{3}$. First, if $\ell = 2$ then $\psi = \hat{X}_{32}$. By Lemmata 4.1 and 4.3, we have $\psi|_{SU_3(q)} = X_{32}|_{SU_3(q)} = \chi_{q^3 + 1}^{(q^2 - 1)} \in IBr_2(SU_3(q))$. Next, if $\ell = 3$ then $\psi = \hat{X}_{32} - \hat{1}_G$.

By Lemma 4.1, we have $\psi|_{SU_3(q)} = \hat{X}_{32}|_{SU_3(q)} - \hat{1}_{SU_3(q)} = \chi_{q^3 + 1}^{(q^2 - 1)} - \hat{1}_{SU_3(q)} = \chi_{q^3 + 1}$ which is an irreducible 3-Brauer character of $SU_3(q)$ (cf. [G, p. 573]). Finally, if $\ell \neq 2, 3$ then $\psi = \hat{X}_{32}$.

By Lemma 4.1, we have $\hat{X}_{32}|_{SU_3(q)} = \chi_{q^3 + 1}^{(q^2 - 1)}$ which is irreducible by [G]. We have shown that when $q \equiv 1 \pmod{3}$, $\psi|_{SU_3(q) : 2}$ is irreducible, as we claim in the item (ii) of Theorem 1.1.

**Case 5.** $M = G_2(q_0)$ with $q = q_0^2$, $3 \nmid q$.

Since $\varphi|_M$ is irreducible, $\varphi(1) \leq \sqrt{|G_2(q_0)|} = \sqrt{q_0^6(q_0^6 - 1)(q_0^2 - 1)} < \sqrt{q_0^7} < \varphi_{2, \ell}(G)$ for every $q \geq 5$. Therefore, by Theorem 3.1, the unique possibility for $\varphi$ is $\psi$. Since $q = q_0^2$, $q \equiv 1 \pmod{3}$.

Recall that $X_{32}(1) = q^3 + 1$ and $|G_2(q_0)| = q_0^6(q_0^6 - 1)(q_0^2 - 1) = q_0^3(q^3 - 1)(q - 1)$. It is easy to see that $(q^3 + 1) \nmid q^3(q^3 - 1)(q - 1)$ for every $q \geq 5$. So $X_{32}|_{G_2(q_0)}$ is irreducible. First we consider the case $\ell = 3$. Then $\psi = \hat{X}_{32} - \hat{1}_G$. Assume that $\psi|_{G_2(q_0)} = \hat{X}_{32}|_{G_2(q_0)} - \hat{1}_{G_2(q_0)}$ is irreducible. Then $X_{32}|_{G_2(q_0)} = \lambda + \mu$ where $\lambda = \hat{1}_{G_2(q_0)}$, $\mu \in \text{Irr}(G_2(q_0))$ and $\mu \in IBr_3(G_2(q_0))$. We then have $\mu(1) = X_{32}(1) - 1 = q^3 - q_0^6$. So $\mu$ is the Steinberg character of $G_2(q_0)$. From [HS1], we know that the reduction modulo 3 of the Steinberg character is reducible, a contradiction. Now we can assume $\ell \neq 3$ and therefore $\psi = \hat{X}_{32}$. From the reducibility of $X_{32}|_{G_2(q_0)}$ as noted above, $\psi|_{G_2(q_0)}$ is reducible. □

5. Small groups

In this section, we mainly use results and notation of [Atlas1] and [Atlas2].
Lemma 5.1. Theorem 1.2 holds in the case \( G = G_2(3), \, M = U_3(3) : 2 \).

Proof. According to [Atlas1, p. 14], we have \( m_{\mathbb{C}}(U_3(3)) = 32 \) and \( m_{\mathbb{C}}(U_3(3) : 2) = 64 \). Thus, if \( \varphi|_M \) is irreducible then \( \varphi(1) \leq 64 \). Inspecting the character tables of \( G_2(3) \) in [Atlas1, p. 60] and [Atlas1, pp. 140, 142, 143], we see that \( \varphi(1) = 14 \) or 64.

Note that \( G_2(3) \) has a unique irreducible complex character of degree 14 which is denoted by \( \chi_2 \) and every reduction modulo \( \ell \neq 3 \) of \( \chi_2 \) is still irreducible. Now we will show that \( \chi_2|_{U_3(3)} = \chi_6 \), which is the unique irreducible character of degree 14 of \( U_3(3) \). Suppose that \( \chi_2|_{U_3(3)} \neq \chi_6 \), then \( \chi_2|_{U_3(3)} \) is reducible and it is the sum of more than one irreducible characters of degree less than 14. Note that \( U_3(3) \) has exactly one conjugacy class of elements of order 6, which is denoted by \( 6A \). When \( \chi_2|_{U_3(3)} \) is the sum of two irreducible characters, then the degrees of these characters are both equal to 7. So \( \chi_2|_{U_3(3)}(6A) = 2 \) or 4. This cannot happen since the value of \( \chi_2 \) on any class of elements of order 6 of \( G_2(3) \) is 1 or \(-2\). If \( \chi_2|_{U_3(3)} \) is the sum of more than two irreducible characters, then \( \chi_2|_{U_3(3)}(6A) \geq 2 \) which cannot happen, neither. So we have \( \chi_2|_{U_3(3)} = \chi_6 \). We also see that every reduction modulo \( \ell \neq 3 \) of \( \chi_6 \) is still irreducible. Hence, if \( \ell \neq 3 \) and \( \varphi \) is the irreducible \( \ell \)-Brauer character of \( G_2(3) \) of degree 14, then \( \varphi|_{U_3(3)} \) is irreducible and so is \( \varphi|_{U_3(3):2} \).

By [Atlas1, p. 14], \( U_3(3) : 2 \) has one irreducible complex character of degree 64 which we denote by \( \chi \). Also, \( G_2(3) \) has two irreducible complex characters of degree 64 that are \( \chi_3 \) and \( \chi_4 \) as denoted in [Atlas1, p. 60]. We will show that \( \chi_3,4|_{U_3(3):2} = \chi \). Note that the two conjugacy classes of \( (U_3(3) : 2) \)-subgroups of \( G_2(3) \) are fused under an outer automorphism \( \tau \) of \( G_2(3) \), which stabilizes each of \( \chi_3 \) and \( \chi_4 \). Hence, without loss we may assume that \( U_3(3) : 2 \) is the one considered in [E, p. 237]. Checking directly, it is easy to see that the values of \( \chi_3 \) and \( \chi_4 \) coincide with those of \( \chi \) at every conjugacy classes except at the classes of elements of order 3. \( U_3(3) : 2 \) has two classes of elements of order 3, \( 3A \) and \( 3B \). Using [E, p. 237] to find the fusion of conjugacy classes of \( U_3(3) \) in \( G_2(3) \) and the values of \( \chi_3 \) and \( \chi_4 \) (which are \( \theta_{12}(k) \) in [E]), we see that the classes \( 3A, 3B \) of \( U_3(3) \) are contained in the classes \( 3A, 3E \) of \( G_2(3) \), respectively. Moreover, \( \chi_3(3A) = \chi_4(3A) = \chi(3A) = -8 \) and \( \chi_3(3E) = \chi_4(3E) = \chi(3B) = -2 \). We have shown that \( \chi_3|_{U_3(3):2} = \chi_4|_{U_3(3):2} = \chi \). By [Atlas2, pp. 140, 142, 143], the reductions modulo \( \ell \neq 3 \) of \( \chi_3 \) as well as \( \chi_4 \) are irreducible. Also, the reduction modulo \( \ell \) of \( \chi \) is irreducible for every \( \ell \neq 3,7 \). Therefore, \( \tilde{\chi}_3|_{U_3(3):2} \) and \( \tilde{\chi}_4|_{U_3(3):2} \) are equal and irreducible for every \( \ell \neq 3,7 \). When \( \ell = 7 \), both \( \tilde{\chi}_3|_{U_3(3):2} \) and \( \tilde{\chi}_4|_{U_3(3):2} \) are reducible since \( m_7(U_3(3) : 2) = 56 \) and \( \tilde{\chi}_3(1) = \tilde{\chi}_4(1) = 64 \). □

Lemma 5.2. Theorem 1.2 holds in the case \( G = G_2(3), \, M = 2^3 \cdot L_3(2) \).

Proof. We have \( |2^3 \cdot L_3(2)| = 1344 \). So \( m_{\mathbb{C}}(2^3 \cdot L_3(2)) \leq \sqrt{1344} < 37 \) and therefore the unique possibility for \( \varphi \) is the reduction modulo \( \ell \neq 3 \) of the character \( \chi_2 \) of degree 14.

When \( \ell \mid |G_2(3)| \), i.e. \( \ell \neq 2, 3, 7, \) and 13, from [KT], we know that \( \tilde{\chi}_2|_{2^3 \cdot L_3(2)} \) is irreducible.

When \( \ell = 2 \), we have \( m_2(2^3 \cdot L_3(2)) = m_2(L_3(2)) \leq \sqrt{168} < 13 \). So \( \tilde{\chi}_2|_{2^3 \cdot L_3(2)} \) is reducible when \( \ell = 2 \). When \( \ell = 13 \), since \( 13 \mid |2^3 \cdot L_3(2)| \), \( \tilde{\chi}_2|_{2^3 \cdot L_3(2)} \) is still irreducible. The last case we need to consider is \( \ell = 7 \). Let \( E = 2^3 \cdot L_3(2) \) which is an elementary abelian group of order \( 2^3 \). Since \( \chi_2|_{E,L_3(2)} \) is irreducible and \( L_3(2) \) acts transitively on \( E \setminus \{1\} \) and \( \text{Irr}(E) \setminus \{1_E\} \), \( \chi_2|_E = 2 \cdot \sum_{\alpha \in \text{Irr}(E) \setminus \{1_E\}} \alpha \). Let \( I \) be the inertia group of \( \alpha \) in \( E \cdot L_3(2) \). By Clifford’s theorem, we have \( \chi_2|_{E,L_3(2)} = \text{Ind}_I^{E \cdot L_3(2)}(\rho) \) for some \( \rho \in \text{Irr}(I) \) and \( \rho|_E = 2\alpha \). We also have
Then we have $|I| = \frac{|E \cdot L_3(2)|}{|\text{Inn}(E) \setminus \{1_E\}|} = 2^6 \cdot 3$, which is not divisible by 7. Thus, the reduction modulo 7 of $\rho$ is irreducible. Hence $\mathcal{Z}_{E \cdot L_3(2)}$ is also irreducible when $\ell = 7$. □

**Lemma 5.3.** Theorem 1.2 holds in the case $G = 3 \cdot G_2(3)$, $M = 3P$ or $3Q$ where $P$, $Q$ are maximal parabolic subgroups of $G_2(3)$.

**Proof.** First, we consider $M = 3P$ where $P$ is one of two maximal parabolic subgroups which is specified in [E, p. 217]. If $\varphi|_{3P}$ is irreducible then $\varphi(1) \leq mC(3P) \leq \sqrt{|(3P)/Z(3P)|} \leq \sqrt{|P|} = \sqrt{11664} = 108$. Inspecting the character tables (both complex and Brauer) of $3 \cdot G_2(3)$ in [Atlas1] and [Atlas2] (note that we only consider faithful characters), we have $\varphi(1) = 27$ and $\varphi$ is actually the reduction modulo $\ell \neq 3$ of any of the four irreducible complex characters of degree 27 of $3 \cdot G_2(3)$. From now on, we denote these characters by $\chi_{24}, \chi_{24}^{\star}$ (corresponding to the line $\chi_{24}$ in [Atlas1, p. 60]) and $\chi_{25}, \chi_{25}^{\star}$ (corresponding to the line $\chi_{25}$ in [Atlas1, p. 60]).

Now we will show that $\chi_{24}|_{3P}$ is irreducible. Note that if $g_1$ and $g_2$ are the pre-images of an element $g \in G_2(3)$ under the natural projection $\pi: 3 \cdot G_2(3) \rightarrow G_2(3)$, then $\chi_{24}(g_1) = \omega \chi_{24}(g_2)$ where $\omega$ is a cubic root of unity. Therefore we have $[\chi_{24}|_{3P}, \chi_{24}|_{3P}]_{3P} = \frac{1}{|P|} \sum_{x \in 3P} \chi_{24}(x) \chi_{24}(x) = \frac{1}{|P|} \sum_{g \in P} \chi_{24}(g) \chi_{24}(g)$, where $g$ is a fixed pre-image for each $g$. We choose $\tilde{g}$ so that the value of $\chi_{24}$ at $\tilde{g}$ is printed in [Atlas1, p. 60]. In [E, p. 217], we have the fusion of conjugacy classes of $P$ in $G_2(3)$. By comparing the orders of centralizers of conjugacy classes of $G_2(3)$ in [E, p. 239] with those in [Atlas1, p. 60], we can find a correspondence between conjugacy classes of $G_2(3)$ in these two papers. The length of each conjugacy class of $P$ can be computed from [E, pp. 217, 218]. All the above information is collected in Table 4. From this table, we see that the value of $\chi_{24}$ is zero at any element $\tilde{g}$ for which the order of $g$ is 3, 6 or 9. We denote by $\chi_{24}(\tilde{X})$ the value $\chi_{24}(\tilde{g})$ for some $g \in X$, where $X$ is a conjugacy class of $P$. Then we have

$$
\sum_{g \in P} \chi_{24}(\tilde{g}) \chi_{24}(\tilde{g}) = |A_1| (\chi_{24}(\tilde{A}_1))^2 + |B_1| (\chi_{24}(\tilde{B}_1))^2 + |B_2| (\chi_{24}(\tilde{B}_2))^2 + |D_1| (\chi_{24}(\tilde{D}_1))^2 + 2|E_2(1)| (\chi_{24}(\tilde{E}_2(1)))^2.
$$

(5.7)

By [E, p. 217], the class $D_1$ of $P$ is contained in the class $D_{11}$ of $G_2(3)$. This class is the class $4A$ or $4B$ according to the notation in [Atlas1, p. 60]. First we assume $D_{11}$ is $4A$. Then by looking at the values of one character of degree 273 of $G_2(3)$ both in [E] and [Atlas1], it is easy to see that $D_2 \subset D_{12} = 12A$ and $E_2(i) \subset E_2 = 8A$. Therefore, formula (5.7) becomes

$$
\sum_{g \in P} \chi_{24}(\tilde{g}) \chi_{24}(\tilde{g}) = 27^2 + 324 \cdot 3^2 + 81 \cdot 3^2 + 486 \cdot (-1)^2 + 972 \cdot 2^2 + 2 \cdot 1458 \cdot (-1)^2
$$

$$
= 11,664.
$$

Next, we assume $D_{11}$ is $4B$. Then $D_{12} = 12B$, $E_2 = 8B$ and formula (5.7) becomes

$$
\sum_{g \in P} \chi_{24}(\tilde{g}) \chi_{24}(\tilde{g}) = 27^2 + 324 \cdot 3^2 + 81 \cdot 3^2 + 486 \cdot 3^2 + 972 \cdot 0^2 + 2 \cdot 1458 \cdot 1^2
$$

$$
= 11,664.
$$
Table 4
Fusion of conjugacy classes of $P$ in $G_2(3)$

<table>
<thead>
<tr>
<th>Fusion in [E]</th>
<th>Corresponding class in [Atlas1]</th>
<th>Length</th>
<th>Values of $\chi_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 \subset A_1$</td>
<td>$1A$</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td>$A_2 \subset A_2$</td>
<td>$1A$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_3 \subset A_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{41} \subset A_{31}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{42} \subset A_{32}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{43} \subset A_{32}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{51} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{52} \subset A_{42}$</td>
<td>$3A, 3B, 3C, 3D, 3E$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$A_{61} \subset A_{31}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{62} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{63} \subset A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{64} \subset A_{41}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{65(t)} \subset A_{32}, A_{41}, A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{66(t)} \subset A_{41}, A_{42}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{71} \subset A_{51}$</td>
<td>$9A, 9B, 9C$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$A_{72} \subset A_{52}$</td>
<td>$9A, 9B, 9C$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$A_{73} \subset A_{53}$</td>
<td>$9A, 9B, 9C$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{11} \subset B_1$</td>
<td>$2A$</td>
<td>324</td>
<td>3</td>
</tr>
<tr>
<td>$B_{12} \subset B_2$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{13} \subset B_3$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{14} \subset B_4$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{15} \subset B_5$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{21} \subset B_1$</td>
<td>$2A$</td>
<td>81</td>
<td>3</td>
</tr>
<tr>
<td>$B_{22} \subset B_2$</td>
<td>$2A$</td>
<td>81</td>
<td>3</td>
</tr>
<tr>
<td>$B_{23} \subset B_3$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{24} \subset B_4$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$B_{25} \subset B_5$</td>
<td>$6A, 6B, 6C, 6D$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$D_1 \subset D_{11}$</td>
<td>$4A, 4B$</td>
<td>486</td>
<td>$-1, 3$</td>
</tr>
<tr>
<td>$D_2 \subset D_{12}$</td>
<td>$12A, 12B$</td>
<td>972</td>
<td>$2, 0$</td>
</tr>
<tr>
<td>$E_{2(i)} \subset E_2$ (two classes)</td>
<td>$8A, 8B$</td>
<td>1458</td>
<td>$-1, 1$</td>
</tr>
</tbody>
</table>

So in any case, we have $[\chi_{24}|_{3}P, \chi_{24}|_{3}.P]_{3} = \frac{1}{|P|} \sum_{g \in P} \chi_{24}(g) \overline{\chi_{24}(g)} = 1$. That means $\chi_{24}|_{3}P$ is irreducible.

Note that $\overline{\chi_{24}}$ is irreducible for every $\ell \neq 3$ by [Atlas2, pp. 140, 142, 143]. We will show that $\overline{\chi_{24}}|_{3}P$ is also irreducible for every $\ell \neq 3$. The structure of $P$ is $[3^5] : 2S_4$. We denote by $O_3$ the maximal normal 3-subgroup (of order $3^5$) of $P$. Then the order of any element in $O_3$ is 1, 3 or 9, and $3. O_3$ is the maximal normal 3-subgroup of $3.P$. Since the values of $\chi_{24}$ is zero at any element $g$ where the order of $g$ is 3 or 9, we have

$$[\chi_{24}|_{3}.O_3, \chi_{24}|_{3}.O_3]_{3}.O_3 = \frac{1}{|O_3|} \sum_{g \in O_3} \chi_{24}(g) \overline{\chi_{24}(g)} = \frac{1}{3^5}27^2 = 3.$$

By Clifford’s theorem, $\chi_{24}|_{3}.O_3 = e \cdot \sum_{i=1}^{l} \theta_i$ where $e = [\chi_{24}|_{3}.O_3, \theta_1]_{3}.O_3$ and $\theta_1, \theta_2, \ldots, \theta_l$ are
the distinct conjugates of \( \theta_1 \) in \( 3.P \). So \( e^2 t = 3 \) and therefore \( e = 1, t = 3 \). Thus, \( \chi_{24}|3.0_3 = \theta_1 + \theta_2 + \theta_3 \). By Lemma 2.6, \( \chi_{24}|3.\phi \) is irreducible when \( \ell \neq 3 \).

Arguing similarly, we also have \( \chi_{25}|3.\phi \) is irreducible and therefore \( \hat{\chi}_{24}|3.\phi \) as well as \( \hat{\chi}_{25}|3.\phi \) are irreducible for every \( \ell \neq 3 \). Note that an outer automorphism \( \tau \) of \( G_2(3) \) sends \( P \) to \( Q \) and fixes \( \{ \chi_{24}, \chi_{24}, \chi_{25}, \chi_{25} \} \). Hence the lemma also holds when \( M = 3.Q \) by Lemma 2.5.

**Lemma 5.4.** *Theorem 1.2 holds in the case \( G = 3 \cdot G_2(3), M = 3.(U_3(3) : 2) \).*

**Proof.** Since the Schur multiplier of \( U_3(3) \) is trivial and \( \mathbb{Z}_3 = Z(3 \cdot G_2(3)) \leq Z(M), 3.(U_3(3) : 2) = 3 \times (U_3(3) : 2) \). So \( m_{\mathbb{C}}(M) = m_{\mathbb{C}}(U_3(3) : 2) = 64 \). Therefore if \( \phi|_M \) is irreducible then \( \phi(1) = 27 \) and \( \phi \) is the reduction modulo \( \ell \neq 3 \) of any of the four irreducible complex characters of degree 27 which we denote by \( \chi_{24}, \chi_{25}, \chi_{24} \), and \( \chi_{25} \) as before.

When \( \ell = 2 \) or \( 7, U_3(3) : 2 \) does not have any irreducible \( \ell \)-Brauer character of degree 27. So we assume that \( \ell \neq 2, 3 \), and 7. That means \( \ell \mid |M| \) and \( \text{IBr}_\ell(M) = \text{Irr}(M) \). Therefore we only need to consider the complex case.

It is obvious that \( \psi_1|_{3 \times (U_3(3) : 2)} \) is irreducible if and only if \( \phi|_{(U_3(3) : 2)} \) is irreducible. Moreover, since \( \phi(1) = 27, \phi|_{U_3(3) : 2} \) is irreducible if and only if \( \phi|_{U_3(3)} \) is irreducible. Now we will show that either \( \chi_{24}|_{U_3(3)} \) or \( \chi_{25}|_{U_3(3)} \) is irreducible and the other is reducible.

From [Atlas1, p. 14], we know that \( U_3(3) \) has the unique irreducible character of degree 27, which is denoted by \( \chi_{10} \). It is easy to see that the classes \( 4A, 4B \) of \( U_3(3) \) are contained in the same class of \( G_2(3) \), which is \( 4A \) or \( 4B \). With no loss we suppose that this class is \( 4A \). Then the class \( 4C \) of \( U_3(3) \) is contained in the class \( 4B \) of \( G_2(3) \). In \( U_3(3) \) we have \( (8A)^2 \subset 4A \), \( (8A)^2 \subset 4B \) and in \( G_2(3) \), \( (8A)^2 \subset 4A \), \( (8B)^2 \subset 4B \). So the classes \( 8A \), \( 8B \) of \( U_3(3) \) are contained in the class \( 8A \) of \( G_2(3) \). Similarly, the classes \( 12A \), \( 12B \) of \( U_3(3) \) are contained in the class \( 12A \) of \( G_2(3) \). Comparing the values of \( \chi_{24}|_{U_3(3)} \) as well as \( \chi_{25}|_{U_3(3)} \) with those of \( \chi_{10} \) on every conjugacy classes of \( U_3(3) \), we see that \( \chi_{25}|_{U_3(3)} = \chi_{10} \) and \( \chi_{24}|_{U_3(3)} \) is reducible.

Note that \( G_2(3) \) has two non-conjugate maximal subgroups which are isomorphic to \( U_3(3) : 2 \). We denote these groups by \( M_1 \) and \( M_2 \). Suppose that \( \chi_{25}|_{M_1} \) is irreducible and \( \chi_{24}|_{M_1} \) is reducible. Let \( \tau \) be an automorphism of \( G_2(3) \) such that \( \tau(M_1) = M_2 \). Then \( \tau(M_2) = M_1, \chi_{24} \circ \tau = \chi_{25} \) and \( \chi_{25} \circ \tau = \chi_{24} \). By Lemma 2.5, \( \chi_{25}|_{M_2} \) is reducible and \( \chi_{24}|_{M_2} \) is irreducible.

**Lemma 5.5.** *Theorem 1.2 holds in the case \( G = 3 \cdot G_2(3), M = 3.(L_3(3) : 2) \) or \( 3.(L_2(8) : 3) \).*

**Proof.** This lemma is proved similarly as the previous lemma.

**Lemma 5.6.** *Theorem 1.2 holds in the case \( G = G_2(4), M = U_3(4) : 2 \).*

**Proof.** We have \( m_{\mathbb{C}}(U_3(4)) = 75 \) and \( m_{\mathbb{C}}(U_3(4) : 2) = 150 \). So if \( \psi|_M \) is irreducible then \( \phi(1) \leq 150 \). Inspecting the character table of \( G_2(4) \), we see that \( \phi(1) = 64, 78 \) when \( \ell = 3 \) or \( \phi(1) = 65, 78 \) when \( \ell \neq 2, 3 \). Using Lemma 4.1 and arguing as in Case 4 of the proof of Theorem 1.1, we see that the restriction of the character of smallest degree of \( G_2(4) \) to \( U_3(4) \) is irreducible.

It remains to consider the case that \( \phi \) is the reduction modulo \( \ell \neq 2 \) of the character \( \chi_3 \) (as denoted in [Atlas1, p. 98]) of degree 78. Because \( U_3(4) : 2 \) has no complex irreducible character of degree 78, \( \chi_3|_{U_3(4) : 2} \) is reducible and so is \( \hat{\chi}_3|_{U_3(4) : 2} \).
Lemma 5.7. Theorem 1.2 holds in the case $G = G_2(4)$, $M = J_2$.

Proof. Comparing the degrees of irreducible $\ell$-Brauer characters of $G_2(4)$ with those of $J_2$, we see that if $\varphi \mid J_2$ is irreducible then $\varphi(1)$ only can be any of the two irreducible characters of degree 300 with $\ell \neq 2, 3$ and 7. When $\ell = 0$, these two complex characters are denoted by $\chi_4$ and $\chi_5$ in [Atlas1, p. 98]. First, we will show that $\chi_4 \mid J_2$ is actually irreducible. More precisely, $\chi_4 \mid J_2 = \chi_20$, where $\chi_20$ is the unique irreducible complex character of $J_2$ of degree 300.

It is easy to see that the values of $\chi_4$ and $\chi_20$ are the same at conjugacy classes of elements of order 5, 7, 8, 10, and 15. The unique class 12A of elements of order 12 in $J_2$ is real. So it is contained in a real class of $G_2(4)$. Therefore it is contained in class 12A of $G_2(4)$ and we have $\chi_4(12A) = \chi_20(12A) = 1$. We see that $(12A)^3 = 4A$ in both $G_2(4)$ and $J_2$. Therefore the class 4A of $J_2$ is contained in the class 4A of $G_2(4)$ and we also have $\chi_4(4A) = \chi_20(4A) = 4$. Now we consider classes of elements of order 2, 3 and 6. Since $J_2$ is a subgroup of $G_2(4)$, either $2 \cdot J_2$ (the universal cover of $J_2$) or $2 \times J_2$ is a subgroup of $2 \cdot G_2(4)$. Note that $\varphi(C(2 \cdot G_2(4))) = 12$ and $\varphi(C(J_2)) = 4(2 \times J_2) = 14$. So $2 \times J_2$ cannot be a subgroup of $2 \cdot G_2(4)$ and therefore $2 \cdot J_2$ is a subgroup of $G_2(4)$. From [Atlas1], the class 2A of $G_2(4)$ lifts to two involution classes of $2 \cdot G_2(4)$ and the class 2B of $G_2(4)$ lifts to a class of elements of order 4 of $2 \cdot G_2(4)$. In the same way, the class 2A of $J_2$ lifts to two involution classes of $2 \cdot J_2$ and the class 2B of $J_2$ lifts to a class of elements of order 4 of $2 \cdot J_2$. These imply that the classes 2A and 2B of $J_2$ are contained in the classes 2A and 2B of $G_2(4)$, respectively. Again, we have $\chi_4(2A) = \chi_20(2A) = -20$ and $\chi_4(2B) = \chi_20(2B) = 0$. Using similar arguments, we also can show that the classes 6A and 6B of $J_2$ are contained in the classes 6A and 6B of $G_2(4)$, respectively. Again, $\chi_4(6A) = \chi_20(6A) = 1$ and $\chi_4(6B) = \chi_20(6B) = 0$. In both $G_2(4)$ and $J_2$, we have $(6A)^2 = 3A$ and $(6B)^2 = 3B$. That means the classes 3A and 3B of $J_2$ are contained in the classes 3A and 3B of $G_2(4)$, respectively. One more time, $\chi_4(3A) = \chi_20(3A) = -15$ and $\chi_4(3B) = \chi_20(3B) = 0$. We have shown that the values of $\chi_4$ and $\chi_20$ agree at all conjugacy classes of $J_2$. Therefore $\chi_4 \mid J_2 = \chi_20$. Note that $\chi_5 = \chi_4$ and all irreducible characters of $J_2$ are real. Therefore we also have $\chi_5 \mid J_2 = \chi_20$.

When $\ell \neq 2, 3$, and 7, the reductions modulo $\ell$ of $\chi_4$, $\chi_5$ and $\chi_20$ are still irreducible. Thus, $\chi_4 \mid J_2$ and $\chi_5 \mid J_2$ are irreducible for every $\ell \neq 2, 3$, and 7. □

Lemma 5.8. Theorem 1.2 holds in the case $G = 2 \cdot G_2(4)$, $M = 2.P$.

Proof. We have $m_C(2.P) \leq \sqrt{[2.P/Z(2.P)] \leq \sqrt{|P| = \sqrt{184,320} < 430}$. Therefore if $\varphi \mid M$ is irreducible then $\varphi(1) < 430$. There are five cases to consider:

• $\varphi(1) = 12$ when $\ell \neq 2$. Then $\varphi$ is the reduction modulo $\ell \neq 2$ of the unique irreducible complex character of degree 12 of $2 \cdot G_2(4)$. Throughout the proof of this lemma we denote this character by $\chi$. Now we will show that $\chi \mid 2.P$ is irreducible.

Note that if $g_1$ and $g_2$ are the pre-images of an element $g \in G_2(4)$ under the natural projection $\pi : 2 \cdot G_2(4) \to G_2(4)$, then $\chi(g_1) = \pm \chi(g_2)$. Therefore $[\chi_2.P : \chi_2.P]_2.P = \frac{1}{|P|} \sum_{x \in 2.P} \chi(x) \chi(x) = \frac{1}{|P|} \sum_{x \in P} \chi(g) \chi(\bar{g})$, where $\bar{g}$ is a pre-image of $g$. We choose $\bar{g}$ so that the value of $\chi$ at $\bar{g}$ is printed in [Atlas1, p. 98]. In [EY, p. 357], we have the fusion of conjugacy classes of $P$ in $G_2(4)$. By comparing the orders of centralizers of conjugacy classes of $G_2(4)$ in [EY, p. 364] with those in [Atlas1, p. 98] and looking at the values of irreducible characters of degrees 65, 78, we can find a correspondence between conjugacy classes of $G_2(4)$
Therefore, \( F \) is irreducible.

Next, we show that \( \chi \), \( P \) is also irreducible for every \( \ell \neq 2 \). Set \( O_2 = 2^{2+8} \) to be the maximal normal 2-subgroup of \( P \). Then \( O_2 \) is a union of conjugacy classes of \( P \). Note that \( O_2 \) is a 2-group of exponent 4 and so the orders of elements in \( O_2 \) are 1, 2 or 4. Hence \( O_2 \) is either \( A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \) or \( A_0 \cup A_1 \cup A_2 \cup A_61 \). If the latter case happens then \( [\chi|_{2.02}, \chi|_{2.02}] = \frac{1}{|O_2|} \sum_{g \in O_2} \chi(\overline{g}) \overline{\chi(\overline{g})} = (12^2 + 3 \cdot (-4)^2 + 60 \cdot (-4)^2 + 120 \cdot (-4)^2 + 1440 \cdot (-4)^2) \)

\[
+ 5760 \cdot 2^2 + 320 \cdot (-6)^2 + 960 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3840 \cdot 2^2 + 3 \cdot (-3)^2 + 2 \cdot 9216 \cdot 1^2
\]

\[= 184,320.\]

Therefore, \( [\chi|_{2.P}, \chi|_{2.P}] = \frac{1}{|P|} \sum_{g \in P} \chi(\overline{g}) \overline{\chi(\overline{g})} = 1 \), which implies that \( \chi \) is irreducible.

<table>
<thead>
<tr>
<th>Fusion in ( E )</th>
<th>Corresponding class ([\text{Atlas1}])</th>
<th>Length</th>
<th>Value of ( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_0 \subset A_0 )</td>
<td>1A</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>( A_1 \subset A_1 )</td>
<td>2A</td>
<td>3</td>
<td>-4</td>
</tr>
<tr>
<td>( A_2 \subset A_1 )</td>
<td>2A</td>
<td>60</td>
<td>-4</td>
</tr>
<tr>
<td>( A_3 \subset A_2 )</td>
<td>2B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>( A_{41} \subset A_{31} )</td>
<td>4A</td>
<td>120</td>
<td>-4</td>
</tr>
<tr>
<td>( A_{42} \subset A_{32} )</td>
<td>4C</td>
<td>360</td>
<td>0</td>
</tr>
<tr>
<td>( A_5 \subset A_4 )</td>
<td>4B</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>( A_{61} \subset A_2 )</td>
<td>2B</td>
<td>960</td>
<td>0</td>
</tr>
<tr>
<td>( A_{62} \subset A_{31} )</td>
<td>4A</td>
<td>1440</td>
<td>-4</td>
</tr>
<tr>
<td>( A_{63} \subset A_{32} )</td>
<td>4C</td>
<td>1440</td>
<td>0</td>
</tr>
<tr>
<td>( A_{71} \subset A_{51} )</td>
<td>8A</td>
<td>5760</td>
<td>0</td>
</tr>
<tr>
<td>( A_{72} \subset A_{52} )</td>
<td>8B</td>
<td>5760</td>
<td>2</td>
</tr>
<tr>
<td>( B_0 \subset B_0 )</td>
<td>3A</td>
<td>320</td>
<td>-6</td>
</tr>
<tr>
<td>( B_1 \subset B_1 )</td>
<td>6A</td>
<td>960</td>
<td>2</td>
</tr>
<tr>
<td>([B_2] \subset B_1 )</td>
<td>6A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>([B_3] \subset B_1 )</td>
<td>6A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>( B_2(0) \subset B_2(0) )</td>
<td>12A</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>( B_2(i) \subset B_2(i) (i = 1, 2) )</td>
<td>12B, 12C</td>
<td>3840</td>
<td>0</td>
</tr>
<tr>
<td>( C_{31}(i) \subset C_{21} ) (two classes)</td>
<td>3B</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>( C_{32}(i) \subset C_{22} ) (two classes)</td>
<td>6B</td>
<td>15,360</td>
<td>0</td>
</tr>
<tr>
<td>( C_{41}(i) \subset C_{21} ) (two classes)</td>
<td>3B</td>
<td>1024</td>
<td>0</td>
</tr>
<tr>
<td>( C_{42}(i) \subset C_{22} ) (two classes)</td>
<td>6B</td>
<td>15,360</td>
<td>0</td>
</tr>
<tr>
<td>( D_{11}(i) \subset D_{11}(i) ) (two classes)</td>
<td>5C, 5D</td>
<td>3072</td>
<td>-3</td>
</tr>
<tr>
<td>( D_{12}(i) \subset D_{12}(i) ) (two classes)</td>
<td>10A, 10B</td>
<td>9216</td>
<td>1</td>
</tr>
<tr>
<td>( E(i) \subset E_1(i) ) (four classes)</td>
<td>15C, 15D</td>
<td>12,288</td>
<td>0</td>
</tr>
</tbody>
</table>
\( \theta_1, \theta_2, \ldots, \theta_t \) are the distinct conjugates of \( \theta_1 \) in \( 2.P \). So \( e^2t = 3 \) and therefore \( e = 1, t = 3 \). Thus \( \chi|_{2.O_2} = \theta_1 + \theta_2 + \theta_3 \). By Lemma 2.6, \( \hat{\chi}|_{2.P} \) is irreducible for every \( \ell \neq 2 \).

- \( \varphi(1) = 104 \) when \( \ell \neq 2, 5 \). In this case, \( \varphi \) is actually the reduction modulo \( \ell \) of a complex irreducible character of degree 104. Note that 104 \( \nmid 368, 640 = |2.P| \) and therefore this case is not an example.

- \( \varphi(1) = 364 \) when \( \ell \neq 2, 3 \). Then \( \varphi \) is the reduction modulo \( \ell \) of a unique faithful irreducible complex character of degree 364. Again, 364 \( \nmid 368, 640 = |2.P| \).

- \( \varphi(1) = 352 \) when \( \ell = 3 \). Denote by \( \chi_{364} \) the unique faithful irreducible complex character of degree 364 of \( 2 \cdot G_2(4) \). Then \( \varphi = \chi_{364} \). Suppose that \( \varphi|_{2.P} \) is irreducible. Because \( \chi_{364}|_{2.P} \) is reducible and \( \hat{\chi}|_{2.P} \) is irreducible, \( \chi_{364}|_{2.P} = \lambda + \mu \) where \( \lambda \) and \( \mu \) are irreducible complex characters of \( 2.P \) such that \( \lambda = \hat{\chi}|_{2.P} \) and \( \mu \in \text{Irr}(2.P) \). So \( \mu \in \text{Irr}(2.P) \) and \( \mu(1) = 352 \) which leads to a contradiction since 352 \( \nmid |2.P| \).

- \( \varphi(1) = 92 \) when \( \ell = 5 \). Denote by \( \chi_{104} \) one of the two complex irreducible characters of degree 104 of \( 2 \cdot G_2(4) \). Then \( \varphi = \chi_{104} - \hat{\chi} \). Suppose that \( \varphi|_{2.P} \) is irreducible. Because \( \chi_{104}|_{2.P} \) is reducible and \( \hat{\chi}|_{2.P} \) is irreducible, \( \chi_{104}|_{2.P} = \lambda + \mu \) where \( \lambda \) and \( \mu \) are irreducible complex characters of \( 2.P \) such that \( \lambda = \hat{\chi}|_{2.P} \) and \( \mu \in \text{Irr}(2.P) \). So \( \mu \in \text{Irr}(2.P) \) and \( \mu(1) = 92 \). Again, we get a contradiction since \( 92 \nmid |2.P| \). \( \square \)

**Lemma 5.9.** Theorem 1.2 holds in the case \( G = 2 \cdot G_2(4), M = 2.Q \).

**Proof.** By similar arguments as in Lemma 5.8, we only need to show that \( \hat{\chi}|_{2.Q} \) is irreducible for every \( \ell \neq 2 \), where \( \chi \) is the unique irreducible character of degree 12 of \( G \). First, let us prove that \( \chi|_{2.Q} \) is irreducible.

We have \( [\chi|_{2.Q}, \chi|_{2.Q}]_{2.Q} = \frac{1}{|Q|} \sum_{x \in Q} \chi(x)\overline{\chi(x)} = \frac{1}{|Q|} \sum_{g \in Q} \chi(\bar{g})\overline{\chi(g)} \), where \( \bar{g} \) is a pre-image of \( g \) under \( \pi \). We choose \( \bar{g} \) so that the value of \( \chi \) at \( \bar{g} \) is printed in [Atlas1, p. 98]. In [EY, p. 361], we have the fusion of conjugacy classes of \( Q \) in \( G_2(4) \) and the length of each conjugacy class of \( Q \), which are described in Table 6. From this table, we get

\[
\sum_{g \in Q} \chi(\bar{g})\overline{\chi(g)} = 12^2 + 15 \cdot (-4)^2 + 360 \cdot (-4)^2 + 240 \cdot (4)^2 + 240 \cdot (-4)^2 \\
+ 5760 \cdot 2^2 + 2 \cdot 64 \cdot (-6)^2 + 2 \cdot 960 \cdot 2^2 + 2 \cdot 3840 \cdot 2^2 \\
+ 2 \cdot 3840 \cdot 2^2 + 2 \cdot 3072 \cdot 2^2 + 4 \cdot 12,288 \cdot (-1)^2 \\
= 184,320.
\]

Therefore, \( [\chi|_{2.Q}, \chi|_{2.Q}]_{2.Q} = \frac{1}{|Q|} \sum_{g \in Q} \chi(\bar{g})\overline{\chi(g)} = 1 \), which implies that \( \chi|_{2.Q} \) is irreducible.

Next, we show that \( \hat{\chi}|_{2.Q} \) is also irreducible. Set \( O_2 = 2^{4+6} \) to be the maximal normal 2-subgroup of \( Q \). Since \( \chi|_{2.Q} \) is irreducible, by Clifford’s theorem, \( \chi|_{2.O_2} = \varepsilon \cdot \sum_{i=1}^t \theta_i \), where \( \varepsilon = [\chi|_{2.O_2}, \chi|_{2.O_2}]_{2.O_2} \) and \( \theta_1, \theta_2, \ldots, \theta_t \) are the distinct conjugates of \( \theta_1 \) in \( 2.Q \). We have \( 12 = \chi(1) = \varepsilon \theta_1(1) \). Note that \( O_2 \) is a 2-group of exponent 4 and so \( \theta_1(1) \in \{1, 2, 4\} \). Therefore \( \varepsilon \in \{3, 6, 12\} \). Set \( m = e^2t = [\chi|_{2.O_2}, \chi|_{2.O_2}]_{2.O_2} = \frac{1}{|O_2|} \sum_{g \in O_2} \chi(\bar{g})\overline{\chi(g)} \). From the above table, we see that \( \chi(\bar{g}) = 0 \) or \( -4 \) for every element \( g \) of order 2 or 4. Therefore \( m = \frac{1}{|O_2|} \sum_{g \in O_2} \chi(\bar{g})\overline{\chi(g)} = \frac{1}{1024} (12^2 + n \cdot (4)^2) \) where \( n \) is the sum of the lengths of conjugacy classes of \( O_2 \) at which the values of \( \chi \) is \( -4 \). Then \( n = (1024 \cdot m - 144)/16 \). We also have \( n \leq 15 + 360 + 240 + 240 = 855 \) and \( 5 \mid n \). This implies that \( m \leq 13 \). Because \( \varepsilon \in \{3, 6, 12\} \),

\( \varepsilon \theta_1(1) \neq 0 \), and \( \varepsilon \neq 0 \), since \( \chi(\bar{g}) \neq 0 \), we get \( m \neq 0 \). Therefore \( \chi|_{2.O_2} \) is irreducible. Hence \( \hat{\chi}|_{2.Q} = \chi|_{2.O_2} \) is irreducible.
Table 6
Fusion of conjugacy classes of $Q$ in $G_2(4)$

<table>
<thead>
<tr>
<th>Fusion in $[EY]$</th>
<th>Corresponding class $[\text{Atlas1}]$</th>
<th>Length</th>
<th>Value of $\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0 \subset A_0$</td>
<td>$1A$</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>$A_1 \subset A_1$</td>
<td>$2A$</td>
<td>15</td>
<td>$-4$</td>
</tr>
<tr>
<td>$A_2 \subset A_2$</td>
<td>$2B$</td>
<td>48</td>
<td>0</td>
</tr>
<tr>
<td>$A_{31} \subset A_2$</td>
<td>$2B$</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{32} \subset A_{31}$</td>
<td>$4A$</td>
<td>360</td>
<td>$-4$</td>
</tr>
<tr>
<td>$A_{33} \subset A_{32}$</td>
<td>$4C$</td>
<td>360</td>
<td>0</td>
</tr>
<tr>
<td>$A_{41} \subset A_1$</td>
<td>$2A$</td>
<td>240</td>
<td>$-4$</td>
</tr>
<tr>
<td>$A_{42}(0) \subset A_{31}$</td>
<td>$4A$</td>
<td>240</td>
<td>$-4$</td>
</tr>
<tr>
<td>$A_{42}(i) \subset A_4$ $(i = 1, 2)$</td>
<td>$4B$</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>$A_{5}(i) \subset A_2, A_{32}, A_4$ (four classes)</td>
<td>$2B, 4C, 4B$</td>
<td>720</td>
<td>0</td>
</tr>
<tr>
<td>$A_{61} \subset A_{51}$</td>
<td>$8A$</td>
<td>5760</td>
<td>0</td>
</tr>
<tr>
<td>$A_{62} \subset A_{52}$</td>
<td>$8B$</td>
<td>5760</td>
<td>2</td>
</tr>
<tr>
<td>$B_0(i) \subset B_0$ $(i = 1, 2)$</td>
<td>$3A$</td>
<td>64</td>
<td>$-6$</td>
</tr>
<tr>
<td>$B_1(i) \subset B_1$ $(i = 1, 2)$</td>
<td>$6A$</td>
<td>960</td>
<td>2</td>
</tr>
<tr>
<td>$B_2(i) \subset B_1$ $(i = 1, 2)$</td>
<td>$6A$</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_3(i, 0) \subset B_2(0)$ $(i = 1, 2)$</td>
<td>$12A$</td>
<td>3840</td>
<td>2</td>
</tr>
<tr>
<td>$B_3(i, j) \subset B_2(j)$ $(i = 1, 2; j = 1, 2)$</td>
<td>$12B, 12C$</td>
<td>3840</td>
<td>0</td>
</tr>
<tr>
<td>$C_{21} \subset C_{21}$</td>
<td>$3B$</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>$C_{22} \subset C_{22}$</td>
<td>$6B$</td>
<td>15,360</td>
<td>0</td>
</tr>
<tr>
<td>$C_{31}(i) \subset C_{21}$ (two classes)</td>
<td>$3B$</td>
<td>5120</td>
<td>0</td>
</tr>
<tr>
<td>$C_{32}(i) \subset C_{22}$ (two classes)</td>
<td>$6B$</td>
<td>15,360</td>
<td>0</td>
</tr>
<tr>
<td>$D_{11}(i) \subset D_{21}(i)$ (two classes)</td>
<td>$5A, 5B$</td>
<td>3072</td>
<td>2</td>
</tr>
<tr>
<td>$D_{12}(i) \subset D_{22}(i)$ (two classes)</td>
<td>$10C, 10D$</td>
<td>9216</td>
<td>0</td>
</tr>
<tr>
<td>$E(i) \subset E_2(i)$ (four classes)</td>
<td>$15A, 15B$</td>
<td>12,288</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

now we see that $m = e^2t \in \{3, 6, 9, 12\}$. If $m = 3$, resp. 9, 12, then $n = 183$, resp. 556, 759, which are coprime to 5. So $m = 6$ and then $n = 375$ which is the sum of lengths of classes $A_1$ and $A_{32}$. We have shown that $e^2t = 6$ and therefore $e = 1$, $t = 6$. Therefore $\chi|_{\ell O_2} = \sum_{i=1}^{6} \theta_i$, which implies that $\hat{\chi}|_{\ell O}$ is irreducible for every $\ell \neq 2$ by Lemma 2.6. □

**Lemma 5.10.** Theorem 1.2 holds in the case $G = 2 \cdot G_2(4)$, $M = 2.(U_3(4) : 2)$ or $M = 2.(SL_3(4) : 2)$.

**Proof.** We give the details of the proof in the case $M = 2.(U_3(4) : 2)$. Note that the Schur multiplier of $U_3(4)$ is trivial. So $M = 2.(U_3(4) : 2) = (2 \times U_3(4)).2$. Therefore $m_{\ell}(M) \leq 2m_{\ell}(2 \times U_3(4)) = 150$ by [Atlas1, p. 30]. Hence, if $\varphi|M$ is irreducible then $\varphi(1) \leq 150$. From the character table of $2 \cdot G_2(4)$, we have $\varphi(1) = 12$ when $\ell \neq 2$, $\varphi(1) = 104$ when $\ell \neq 2, 5$ or $\varphi(1) = 92$ when $\ell = 5$.

- $\varphi(1) = 92$ when $\ell = 5$. Inspecting the 5-Brauer character table of $U_3(4)$ in [Atlas2, p. 72], we see that $M$ does not have any irreducible 5-Brauer character of degree 92. So this case is not an example.
- $\varphi(1) = 12$ when $\ell \neq 2$. In this case, $\varphi$ is the reduction modulo $\ell \neq 2$ of the unique irreducible complex character $\chi$ of degree 12 of $2 \cdot G_2(4)$. Inspecting the character tables of $U_3(4)$, we have $\delta_{\ell}(U_3(4)) = 12$ for every $\ell \neq 2$. It follows that $\hat{\chi}|_{U_3(4)}$ is irreducible and so is $\hat{\chi}|_{M}$.


• \( \varphi(1) = 104 \) when \( \ell \neq 2, 5 \). In this case, \( \varphi \) is the reduction modulo \( \ell \) of any of the two faithful irreducible complex characters of degree 104 of \( 2 \cdot G_2(4) \), which are denoted by \( \chi_{34} \) and \( \chi_{35} \) as in [Atlas1, p. 98]. First, we will show that \( \chi_{34}|_M \) is irreducible.

From Lemma 5.6, we know that the restriction of the unique irreducible character \( \chi_2 \) of degree 65 of \( G_2(4) \) to \( U(3) \) is irreducible and equal to the unique rational irreducible character of degree 65 of \( U(3) \). By looking at the values of these characters, we see that the involution class 2A of \( U(3) \) is contained in the class 2A of \( G_2(4) \) and the class 5E of \( U(3) \) is contained in the class 5A or 5B of \( G_2(4) \). With no loss we assume that this class is 5A.

Now suppose that \( \chi_{34}|_U(3) \) contains the unique irreducible character of degree 64 of \( U(3) \). Since \( \chi_{34}(2A) = 8 \), by looking at the values of irreducible characters of \( U(3) \) at the involution class 2A, we see that the other irreducible constituents of \( \chi_{34}|_U(3) \) are of degrees 1 and 39. But then we get a contradiction by looking at the values of these characters at the class 5E of \( U(3) \).

So \( \chi_{34}|_U(3) \) does not contain the irreducible character of degree 64 of \( U(3) \). Inspecting the character table of \( U(3) \) in [Atlas1, p. 30], we see that \( U(3) \) has four irreducible characters of degree 52, which are denoted by \( \chi_9, \chi_{10}, \chi_{11}, \) and \( \chi_{12} \). Note that \( \chi_{10} = \overline{\chi_9} \) and \( \chi_{12} = \overline{\chi_{11}} \).

We also see that the values of irreducible characters (of degrees different from 64) of \( U(3) \) at the class 5E are: 1, 2, \(-b_5, b_5 + 1, -b_5 + 1, b_5 + 2, b_5, -b_5 - 1, 0 \) where \( b_5 = \frac{1}{2}(-1 + \sqrt{5}) \). Moreover, this value is \( b_5 \) if and only if the character is \( \chi_9 \) or \( \chi_{10} \). Note that \( \chi_{34}(5A) = 2b_5 \).

Now we suppose that \( 2b_5 = x_1 \cdot 1 + x_2 \cdot 2 + x_3 \cdot (-b_5) + x_4 \cdot (b_5 + 1) + x_5 \cdot (-b_5 + 1) + x_6 \cdot (b_5 + 2) + x_7 \cdot b_5 + x_8 \cdot (-b_5 - 1) \) where \( x_i, 8 (i = 1, 2, \ldots, 8) \) are nonnegative integer numbers. Then \( x_1 + 2x_2 + x_4 + x_5 + 2x_6 - x_8 = 0 \) and \( -x_3 + x_4 - x_5 + x_6 + x_7 - x_8 = 2 \). These equations imply that \( 2 = -x_3 + x_4 - x_5 + x_6 + x_7 - (x_1 + 2x_2 + x_4 + x_5 + 2x_6) = -x_1 - 2x_2 - 2x_5 + x_7 \). Therefore \( x_7 \geq 2 \).

In other words, there are at least two irreducible constituents of degree 52 in \( \chi_{34}|_U(3) \). Note that \( \chi_{34}(1) = 104 \). So \( \chi_{34}|_{U(3)} \) must be the sum of two irreducible characters of degree 52. Since \( \chi_{34} \) is real, it is easy to see that \( \chi_{34}|_{U(3)} = \chi_9 + \chi_{10} \).

Notice that \( M = (2 \times U(3)), 2 \) and the “2” is an outer automorphism of \( U(3) \) that fuses \( \chi_9 \) and \( \chi_{10} \). So \( \chi_{34}|M \) is irreducible.

It is easy to see that \( \chi_{35} = * \circ \chi_{34} \) and \( \chi_{11} + \chi_{12} = * \circ (\chi_9 + \chi_{10}) \) where the operator * is the algebraic conjugation: \( r + s\sqrt{5} \mapsto r - s\sqrt{5} \) for \( r, s \in \mathbb{Q} \). Since \( \chi_{34}|_{U(3)} = \chi_9 + \chi_{10}, \chi_{35}|_{U(3)} = \chi_{11} + \chi_{12}, \) which implies that \( \chi_{35}|M \) is irreducible.

When \( \ell \neq 2, 5 \), the reductions modulo \( \ell \) of \( \chi_{34} \) and \( \chi_{35} \) are still irreducible. Arguing similarly as above, we have \( \chi_{34}|_M \) as well as \( \chi_{35}|_M \) are irreducible.

**Proof of Theorem 1.2.** (i) According to [Atlas1, p. 61], if \( M \) is a maximal subgroup of \( G_2(3) \) then \( M \) is \( G_2(3) \)-conjugate to one of the following groups:

1. \( P = [3^5] : 2S_4, Q = [3^5] : 2S_4 \), maximal parabolic subgroups,
2. \( U(3) : 2 \), two non-conjugate subgroups,
3. \( L_3(3) : 2 \), two non-conjugate subgroups,
4. \( L_2(8) : 3 \),
5. \( 2^3 : L_3(2) \),
6. \( L_2(13) \),
7. \( [2^5] : 3^2.2 \).

By Lemmata 5.1–5.2, we need to consider the following cases:
(1) $M = P$ or $Q$. Since the structure of $P$ as well as $Q$ is $[3^5] \cdot 2S_4$, they are solvable. It is well known that every Brauer character of $M$ is liftable to a complex character. Therefore the degree of every irreducible Brauer character of $M$ divides $|P| = |Q| = 11,664$ and less than $\sqrt{11,664} = 108$. Checking both the complex and Brauer character tables of $G_2(3)$, we see that there is no character satisfying these conditions.

(3) $M = L_3(3) : 2$. From [Atlas1, p. 13], we have $m_C(L_3(3) : 2) = 52$. So if $\varphi|_M$ is irreducible then $\varphi(1) \leq 52$ and therefore $\varphi(1) = 14$. That means $\varphi$ is the reduction modulo $\ell \neq 3$ of the unique irreducible complex character of degree 14, which is denoted by $\hat{\chi}_2$ in [Atlas1, p. 60]. Since $14 \nmid |L_3(3) : 2| = 11,232$, $\chi_2|_{L_3(3) : 2}$ as well as $\hat{\chi}_2|_{L_3(3) : 2}$ are reducible for every $\ell \neq 3$.

The cases $M = L_2(8) : 3$, $L_2(13)$, and $[2^5] : 3^2.2$ are treated similarly.

(ii) In this part, we only consider faithful irreducible characters of $3 \cdot G_2(3)$. They are characters which are not inflated from irreducible characters of $G_2(3)$. By Lemma 2.7, a maximal subgroup of $3 \cdot G_2(3)$ is the pre-image of a maximal subgroup of $G_2(3)$ under the natural projection $\pi : 3 \cdot G_2(3) \to G_2(3)$. We denote by $3.X$ the pre-image of $X$ under $\pi$. By Lemmata 5.3–5.5, we need to consider the following cases:

(5) $M = 3.2^3 \cdot L_3(2)$. If $\varphi|_M$ is irreducible then $\varphi(1) \leq m_C(M) \leq |(2^3 \cdot L_3(2))| = \sqrt{1344} < 37$. So $\varphi(1) = 27$ and $\varphi$ is the reduction modulo $\ell \neq 3$ of an irreducible character $\chi$ of degree 27 of $3 \cdot G_2(3)$. Since $27 \nmid |M|$, $\chi|_M$ is reducible and so is $\hat{\chi}|_M$.

(6) $M = 3.L_2(13)$. From [Atlas1, p. 8], the Schur multiplier of $L_2(13)$ has order 2. So $3.L_2(13) = 3 \times L_2(13)$. Hence $m_C(M) = m_C(L_2(13)) = 14$. On the other hand, the degree of any faithful irreducible Brauer character of $3 \cdot G_2(3)$ is at least 27. So we do not have any example in this case.

The case $M = 3.(2^5 : 3^2.2)$ is similar.

(iii) According to [Atlas1, p. 97], if $M$ is a maximal subgroup of $G_2(4)$ then $M$ is $G_2(4)$-conjugate to one of the following groups:

(1) $P = 2^{2+8} : (3 \times A_5)$, $Q = 2^{4+6} : (A_5 \times 3)$, maximal parabolic subgroups,
(2) $U_3(4) : 2$,
(3) $SL_3(4) : 2$,
(4) $U_3(3) : 2$,
(5) $A_5 \times A_5$,
(6) $L_2(13)$,
(7) $J_2$.

By Lemmata 5.6–5.7, we need to consider the following cases:

(1) $M = P$ or $Q$. From [EY], it is easy to see that $m_C(P)$ as well as $m_C(Q)$ are less than 256. So if $\varphi|_M$ is irreducible then $\varphi(1) < 256$. Inspecting the complex and Brauer character tables of $G_2(4)$, we have two possibilities:

- $\ell \neq 2, 3$ and $\varphi$ is the reduction modulo $\ell$ of any of the two characters of degree 65 and 78, which are $\chi_2$ and $\chi_3$ as denoted in [Atlas1, p. 98]. Note that both 65 and 78 do not divide $|P| = |Q| = 184,320$. So both $\hat{\chi}_2|_{P,Q}$ and $\hat{\chi}_3|_{P,Q}$ are reducible.
- $\ell = 3$ and $\varphi(1) = 64$ or 78. The case $\varphi(1) = 78$ cannot happen by a similar reason as above. If $\varphi(1) = 64$ then $\varphi = \hat{\chi}_2 - \overline{1}_{G_2(4)}$. Suppose that the restriction of this character to $P$ is irreducible. Since $\chi_2|_P$ is reducible, $\chi_2|_P = \lambda + \mu$ where $\lambda, \mu \in \text{Irr}(P)$, $\lambda = \overline{1}_P$ and $\hat{\mu} \in \text{IBr}_3(P)$. We then have $\mu(1) = 64$. Inspecting the character table of $P$ in [EY, p. 358], we
see that there is no irreducible character of $P$ of degree 64 and we get a contradiction. The argument for $Q$ is exactly the same.

(3) $M = SL_3(4) : 2$. This case is treated similarly as Case 3 when $q \equiv 1 \pmod{3}$ in the proof of Theorem 1.1.

The cases $M = U_3(3) : 2$, $A_5 \times A_5$, and $L_2(13)$ are similar.

(iv) By Lemma 2.7, a maximal subgroup of $2 \cdot G_2(4)$ is the pre-image of a maximal subgroup of $G_2(4)$ under the natural projection $\pi : 2 \cdot G_2(4) \rightarrow G_2(4)$. We denote by $2X$ the pre-image of $X$ under $\pi$. By Lemmata 5.8–5.10, we need to consider the following cases.

(4) $M = 2.(U_3(3) : 2)$. Since the Schur multiplier of $U_3(3)$ is trivial, $2.U_3(3) = 2 \times U_3(3)$ and $2.(U_3(3) : 2) = (2 \times U_3(3)).2$. So if $\varphi|_M$ is irreducible then $\varphi(1) \leq 2m_3(U_3(3)) = 64$. Therefore $\varphi$ is the reduction modulo $\ell \neq 2$ of $\chi$, the unique irreducible complex character of degree 12 of $2 \cdot G_2(4)$. Assume that $\chi|_M$ is irreducible. Using the character table of $U_3(3)$ in [Atlas1, p. 14], it is easy to see that $\chi|_{U_3(3)} = 2\chi_2$, where $\chi_2$ is the unique irreducible character of degree 6 of $U_3(3)$. Therefore $\chi|_{2.(U_3(3))} = 2(\sigma \otimes \chi_2)$, where $\sigma$ is the nontrivial irreducible character of $Z_2 = Z(2 \cdot G_2(4))$. This and Lemma 2.8 imply that $\chi|_M$ is reducible, a contradiction.

(5) $M = 2.(A_5 \times A_5)$. We denote by $A$ and $B$ the pre-images of the first and second terms $A_5$ (in $A_5 \times A_5$), respectively, under the projection $\pi$. For every $a \in A$, $b \in B$, we have $\pi(\{a, b\}) = [\pi(a), \pi(b)] = 1$. Therefore, $[a, b] \in Z_2$ where $Z_2 = Z(2 \cdot G_2(4)) \leq Z(M)$. This implies that $[[A, B], A] = [B, [A, A]] = 1$. By the 3-subgroup lemma, we have $[[A, A], B] = 1$. Since the Schur multiplier of $A_5$ is 2, $A$ is $2 \times A_5$ or $2.A_5$, the universal cover of $A_5$. If $A = 2.A_5$ then $[A, A] = A$. If $A = 2 \times A_5$ then $[A, A] = A_5$. So, in any case, $[A, B] = 1$ or $A$ centralizes $B$. That means $M = (A \times B)/Z_2$.

We have $m_3(M) \leq \sqrt{|M/Z(M)|} \leq \sqrt{|A_5 \times A_5|} = 60$. Therefore if $\varphi|_M$ is irreducible then $\varphi$ is the reduction modulo $\ell \neq 2$ of $\chi$, the unique complex irreducible character of degree 12 of $2 \cdot G_2(4)$. Now we will show that $\chi|_M$ is reducible.

Suppose $\lambda$ is any irreducible character of $M$ of degree 12. Then we can regard $\lambda$ as an irreducible character of $A \times B$ with $Z_2 \subset \ker \lambda$. Assume that $\lambda = \lambda_A \otimes \lambda_B$ where $\lambda_A \in \text{Irr}(A)$ and $\lambda_B \in \text{Irr}(B)$. There are two possibilities:

- One of $\lambda_A(1)$ and $\lambda_B(1)$ is 2 and the other is 6. With no loss of generality, assume that $\lambda_A(1) = 2$ and $\lambda_B(1) = 6$. From the character tables of $A_5$ and $2.A_5$ in [Atlas1, p. 2], we see that $A_5$ does not have any irreducible character of degree 2 or 6. So $A = 2.A_5$. The value of any irreducible character of degree 2 of $2.A_5$ at any conjugacy class of elements of order 5 is $\frac{1}{5}(-1 \pm \sqrt{5})$. Therefore, $\lambda|_A = 6\lambda_A$ is not rational.

- One of $\lambda_A(1)$ and $\lambda_B(1)$ is 4 and the other is 3. With no loss of generality, assume that $\lambda_A(1) = 4$ and $\lambda_B(1) = 3$. From the character tables of $A_5$ and $2.A_5$ in [Atlas1, p. 2], we see that the value of any irreducible character of degree 3 of $A_5$ or $2.A_5$ at any conjugacy class of elements of order 5 is $\frac{1}{5}(1 \pm \sqrt{5})$. Therefore, $\lambda|_B = 4\lambda_B$ is not rational.

We have shown that any irreducible character of degree 12 of $M$ is not rational. On the other hand, $\chi$ is rational. Hence $\chi|_M$ is reducible and so is $\hat{\chi}|_M$.

The cases $M = 2.L_2(13)$ and $2.J_2$ are treated similarly. $\square$
6. Results for Suzuki and Ree groups

**Theorem 6.1.** Let $G = Sz(q)$ be the Suzuki group where $q = 2^n$, $n$ is odd and $n \geq 3$. Let $\varphi$ be an absolutely irreducible character of $G$ in characteristic $\ell \neq 2$ and $M$ a maximal subgroup of $G$. Assume that $\varphi(1) > 1$. Then $\varphi|_M$ is irreducible if and only if $M$ is $G$-conjugate to the maximal parabolic subgroup of $G$ and $\varphi$ is the reduction modulo $\ell$ of any of the two irreducible complex characters of degree $(q - 1)\sqrt{q/2}$.

**Proof.** According to [Su], if $M$ is a maximal subgroup of $G$, then $M$ is $G$-conjugate to one of the following groups:

1. $P = [q^2], Z_{q-1}$, the maximal parabolic subgroup,
2. $D_{2(q-1)}$,
3. $\mathbb{Z}_{q+\sqrt{q+1}}Z_4$,
4. $\mathbb{Z}_{q-\sqrt{q+1}}Z_4$,
5. $Sz(q_0), q = q_0^\alpha, \alpha$ prime, $q_0 \geq 8$.

By Lemmata 2.1, 2.2 and the irreducibility of $\varphi|_M$, we have $\sqrt{|M|} \geq \ell(G)$, which is larger or equal to $(q - 1)\sqrt{q/2}$ by [T1]. Therefore, $|M| \geq q(q - 1)^2/2.$ This inequality happens if and only if $M$ is the maximal parabolic subgroup of $G$.

The complex character table of $P$ is given in [M, p. 157]. From this table, we have $m_P(P) = (q - 1)\sqrt{q/2}$. Since $\ell(G) \leq \ell(P) \leq m_P(P)$, it follows that $\varphi(1) = (q - 1)\sqrt{q/2}$. Using the notation and results about Brauer trees of the Suzuki group in [B], we have $\varphi = \tilde{\ell}_1$ or $\tilde{\ell}_2$, where $\ell_1$ and $\ell_2$ are the two irreducible complex characters of $Sz(q)$ of degree $(q - 1)\sqrt{q/2}$. Now we will show that the restrictions of these characters to $P$ is indeed irreducible.

If $\ell = 0$, comparing directly the values of characters on conjugacy classes, we see that $\ell_1|_P = \phi_2$ and $\ell_2|_P = \phi_3$, where $\phi_2$ and $\phi_3$ are irreducible characters of $P$ and their values are given in [M, p. 157].

Next, we will show that $\tilde{\phi}_i, i = 2, 3$, are irreducible when $\ell \neq 0, 2$. Consider the element $f$ of order 4 in $P$ which is given in [M, p. 157]. Since $\ell$ is odd and $\text{ord}(f) = 4$, $f$ is an $\ell$-regular element. Assume the contrary that $\tilde{\phi}_2$ is reducible. Then it is the sum of more than one irreducible Brauer characters of $P$ whose degrees are less than $(q - 1)\sqrt{q/2}$. Since $P$ is solvable, every Brauer character of $P$ is liftable to complex characters. Inspecting the complex character table of $P$, we see that the value at the element $f$ of any irreducible character of degree less than $(q - 1)\sqrt{q/2}$ is real. On the other hand, $\phi_2(f)$ is not real, a contradiction. We have shown that $\tilde{\phi}_2$ is irreducible and so is $\tilde{\phi}_3$, as $\tilde{\phi}_3 = \overline{\tilde{\phi}_2}$. □

**Note.** Most of the Schur multipliers of the Suzuki group and the Ree group are trivial except the Schur multiplier of $Sz(8)$, which is an elementary abelian group of order $2^2$. Note that $2^2 = Z(2^2.Sz(8))$ which is not cyclic. Therefore, $2^2.Sz(8)$ does not have any faithful irreducible character. This multiplier $2^2$ has three cyclic quotients of order 2 which are corresponding to groups $2.Sz(8)$, $2'.Sz(8)$ and $2''.Sz(8)$. These groups are permuted by the automorphism group. Let us consider our problem for one of them, say $2.Sz(8)$.

Inspecting the character tables of $2.Sz(8)$, we see that if $\varphi$ is a faithful irreducible character of $2.Sz(8)$, then $\varphi(1) \geq 8$. Therefore, if $\varphi|_M$ is irreducible then $\sqrt{|M/Z(M)|} \geq 8$. So the unique possibility for $M$ is $2.P$, where $P$ is the maximal parabolic subgroup of $Sz(8)$. Moreover, $\varphi(1) \leq$
\[ m_C(2.P) \leq \sqrt{|P|} = \sqrt{448} < 22. \] Hence, we have \( \varphi(1) = 8 \) when \( \ell = 5 \) or \( \varphi(1) = 16 \) when \( \ell = 13 \).

- If \( \varphi(1) = 8 \) when \( \ell = 5 \) then \( \varphi = \varphi_{11} \) as denoted in [Atlas2, p. 64]. Note that \( P = [2^6].7 \) and \( 2.P = 2.(2^6).7 = [2^7].7 \). From [Atlas2, p. 64], the value of \( \varphi_{11} \) at any nontrivial 2-element is 0. Therefore, \( \varphi_{11}([2^7], \varphi_{11}([2^7])|_{2^7}) = 8^2/2^6 = 1 \) and hence \( \varphi_{11}|_{2^7} \) is irreducible. So \( \varphi_{11}|_{2.P} \) is also irreducible.

- If \( \varphi(1) = 16 \) when \( \ell = 13 \) then \( \varphi = \varphi_9 \) as denoted in [Atlas2, p. 65]. Since \( 16 > \sqrt{2^7} \), \( \varphi_9|_{2^7} \) is irreducible. By Lemma 2.8, \( \varphi_9|_{2.P} \) is also irreducible.

So when \( G = 2.Sz(8) \), if \( \varphi \) is faithful, \( \varphi|_M \) is irreducible if and only if \( M = 2.P \) and \( \varphi \) is the unique irreducible 5-Brauer character of degree 8.

**Theorem 6.2.** Let \( G = 2G_2(q) \) be the Ree group where \( q = 3^n \), \( n \) is odd and \( n \geq 3 \). Let \( \varphi \) be an absolutely irreducible character of \( G \) in characteristic \( \ell \neq 3 \) and \( M \) be a maximal subgroup of \( G \). Assume that \( \varphi(1) > 1 \). Then \( \varphi|_M \) is irreducible if and only if \( M \) is \( G \)-conjugate to the maximal parabolic subgroup of \( G \) and \( \varphi \) is the nontrivial constituent (of degree \( q^2 - q \)) of the reduction modulo \( \ell = 2 \) of the unique irreducible complex character of degree \( q^2 - q + 1 \).

**Proof.** According to [K2, p. 181], if \( M \) is a maximal subgroup of \( G \), then \( M \) is \( G \)-conjugate to one of the following groups:

1. \( P = [q^3]: \mathbb{Z}_{q-1} \), the maximal parabolic subgroup,
2. \( 2 \times L_2(q) \), involution centralizer,
3. \( (2^2 \times D_{q+1}^1) : 3 \),
4. \( \mathbb{Z}_{q+\sqrt{q}+1} : \mathbb{Z}_6 \),
5. \( \mathbb{Z}_{q-\sqrt{q}+1} : \mathbb{Z}_6 \),
6. \( 2G_2(q_0) \), \( q = q_0^\alpha \), \( \alpha \) prime.

By Lemmata 2.1, 2.2 and the irreducibility of \( \varphi|_M \), we have \( \sqrt{|M|} \geq \delta \ell(G) \), which is larger or equal to \( q(q-1) \) by [T1]. Therefore, \( |M| \geq q^2(q-1)^2 \). This inequality happens if and only if \( M \) is the maximal parabolic subgroup \( P \). The complex character table of \( P \) is given in [LM, p. 88]. From there, we get \( m_C(P) = q(q-1) \) and therefore \( m_\ell(P) \leq q(q-1) \).

Assume that \( \ell = 0 \) or \( \ell \geq 5 \). Using the results about Brauer trees of \( G \) in [H3], it is easy to check that \( \delta \ell(G) = q^2(q+1) > m_\ell(P) \), which contradicts Lemma 2.1. So it remains to consider \( \ell = 2 \).

We have \( q(q-1) \geq m_\ell(P) \geq \varphi(1) \geq \delta \ell(G) \geq q(q-1) \). Therefore \( \varphi(1) = q(q-1) \). We will check all 2-blocks of \( G \) which are studied in [W] and [LM]. We also use the notation in these papers.

1. \( \xi_9, \xi_{10}, \eta_i^{\pm} \) are of 2-defect 0. Their degrees are all larger than \( q(q-1) \).
2. There is one 2-block of defect 1. All characters in this block are \( \eta_r \) and \( \eta'_r \), whose degrees are \( q^3 + 1 \). By Lemma 2.9, there is a unique irreducible 2-Brauer character of degree \( q^3 + 1 \) in this block.
3. There are several 2-blocks of defect 2. Every character in these blocks has degree \( (q-1)(q^2 - q + 1) \). Applying Lemma 2.9 again, all irreducible 2-Brauer characters in these blocks have degree \( (q-1)(q^2 - q + 1) \).
(4) The principal block and its decomposition matrix is described in [LM]. We have \( \varphi_1(1) = 1 \), \( \varphi_2(1) = q(q - 1) \), \( \varphi_3(1) = (q - 1)(q^2 - \sqrt{q^2(q + 1) + 1}) > q(q - 1) \), \( \varphi_4(1) = \varphi_5(1) = (q - 1)\sqrt{q + 1 - 3\sqrt{q^3}}/2 > q(q - 1) \).

In summary, the unique possibility for \( \varphi \) is \( \varphi = \varphi_2 = \hat{\xi}_2 - \hat{1}_G \) when \( \ell = 2 \), where \( \xi_2 \) is the unique irreducible complex character of degree \( q^2 - q + 1 \) of \( G \). Now we will prove that \( \varphi_2 \mid P \) is indeed irreducible. Suppose that \( \varphi_2 \mid P \) is reducible. Then it is the sum of more than one irreducible 2-Brauer characters of \( P \). These characters are of degrees less than \( q(q - 1) \). Moreover, since \( P \) is solvable, they are liftable. Therefore, their values at the element \( X \) of order 3 (which is a representative of a conjugacy class of \( P \) given in [LM, p. 88]) is positive. On the other hand, \( \varphi_2(X) = -q \) which is negative, a contradiction.

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