THE UNIVERSITY OF AKRON
Mathematics and Computer Science

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Lesson 10: Some Second Degree & Trig Curves

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An Algebra Review in Ten Lessons

I am DPS

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

$a^3a^4 = a^7 \quad (ab)^{10} = a^{10}b^{10}$

$-(ab - (3ab - 4)) = 2ab - 4$

$(ab)^3(a^{-1} + b^{-1}) = (ab)^2(a + b)$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$2x^2 - 3x - 2 = (2x + 1)(x - 2)$

$\frac{1}{2}x + 13 = 0 \implies x = -26$

$G = \{ (x, y) \mid y = f(x) \}$

$f(x) = mx + b$

$y = \sin x$

Last Revision Date: 2/2/2000

Lesson 10: Some Second Degree & Trig Curves

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10. Some Second Degree & Trig Curves

In Lesson 9 we gave an extensive discussion of *first degree curves*; these are equations having a *straight line* as its graph. In this lesson, we begin by exploring selected *second degree curves* followed by a brief survey remarks on *trigonometric functions*.

The general form of a second degree equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

where either $A \neq 0$ or $B \neq 0$. In our lesson, $B = 0$[1], so, in fact we will look at equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

The graph of such an equation may be a parabola, a hyperbola, a circle, or an ellipse (or a *degenerate graph*). We discuss only parabolas and circles leaving the others to a (high school) course in Analytic

[1] $B \neq 0$ induces a rotation of the graph.

Geometry, a course in Precalculus, or a course in Calculus and Analytic Geometry.

Associated with any second degree equation (equation (2)) is a very strong geometry. For example, a parabola has a *focus* and the parabola has a certain *reflection* property that is exploited in the manufacture of flash lights, satellite dishes, parabolic mirrors, etc. These geometric properties will not be covered in these lessons; again they are a topic of study in a course on Analytic Geometry. Instead, we will concentrate on the mechanics: recognition, classification, location, and graphing.

10.1. Parabolas

A parabola is described by a second degree equation (in the variables x and y) in which one variable has power two and the other variable has power one. (In terms of equation (2), this means $A = 0$ or $C = 0$, *but not both.*)

It is important to be able to recognize second degree equations whose graph is a parabola. Study the following examples to acquire an "eye."

Illustration 1. Parabolas.

(a) The x variable has degree 2 and the y variable has degree 1. In this case y is expressible as a *function of* x.

1. Equational Form:

$$x^2 - x + y = 1 \quad 3x^2 - 2x + 3y = 5 \quad y - x + 5x^2 = 0$$

2. Explicit Form: y written (explicitly) as a function of x:

$$y = x^2 \quad y = 1 - x^2 + x \quad y = \tfrac{1}{3}(5x^2 + 2x)$$

(b) The y variable has degree 2 and the x variable has degree 1. In this case x is expressible as a *function of* y.

1. Equational Form:

$$x + y + y^2 = 5 \quad 3x - 6y^2 + y - 8 = 0 \quad y^2 = 2x + 3y$$

2. Explicit Form: x written (explicitly) as a function of y.

$$x = y^2 \quad x = 2y^2 - 5y + \frac{1}{2} \quad x = 4y^2 + 7y - 1$$

■

- x **is of second degree**

An equation in which the x variable has the second degree and the y variable has the first degree is a parabola that either **opens up** or **opens down**.

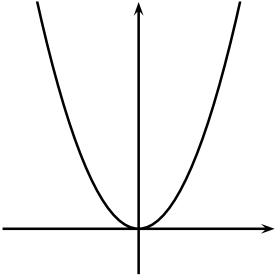


FIGURE 1(a) $y = x^2$
Parabola Opening Up

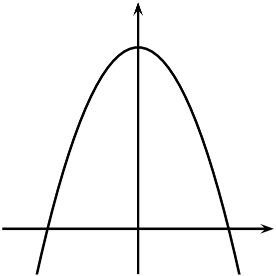


FIGURE 1(b) $y = 4 - x^2$
Parabola Opening Down

Notice that these two parabolas pass the vertical line test; this means that each defines y as a function of x. (The equations are $y = x^2$ and $y = 4 - x^2$.) A parabola has a **vertex** that represents the absolute minimum, see FIGURE 1(a), or the absolute maximum, FIGURE 1(b), of the graph.

The skills associated with working with parabolas are as follows:

- **Recognition:** Recognize the equation as a parabola of this type.
- **Classification:** Classify the parabola as one that opens up or one that opens down.
- **Location:** The parabola is located by calculating its so-called **vertex**.
- **Graphing:** If needed, we need to graph the parabola.

▷ **Recognition.** We've already covered the case for recognition. The important point is that the variable x as degree two and the variable y has degree one.

We shall satisfy ourselves by just presenting a quiz. Passing is 100% so **don't err!**

- Quiz. Click on the green bullet to jump to the quiz on parabolas.

▷ **Classification.** The general form of a parabola that can be written as a function of x is

$$y = ax^2 + bx + c, \quad a \neq 0 \tag{3}$$

We state the following without proof, though the validity of the statements will be made apparent later.

Classification Rule: Consider the parabola

$$y = ax^2 + bx + c, \quad a \neq 0. \tag{4}$$

The parabola **opens up** if $a > 0$ and **opens down** if $a < 0$; i.e., if the coefficient of the second degree term is positive, the parabola opens up, and opens down if the coefficient of the second degree term is negative.

Naturally, you must strive to put your equation in the form of (4) before you can make the correct determination.

This is such a simple principle, let's just quiz you on it.

- Quiz. Click on the green bullet to jump to the quiz on parabolas.

▷ **Location.** A parabola that is a function of x has the form

$$y = ax^2 + bx + c, \quad a \neq 0. \tag{5}$$

Such a parabola opens up (if $a > 0$) or down (if $a < 0$). Another important feature of a parabola is the presence of a *vertex*. The vertex of the parabola in equation (5) can be determined by completing the square of the right-hand side.

Generally, when you complete the square of (5), you obtain an equation of the form

$$y - k = a(x - h)^2, \tag{6}$$

where h and k are constants. It turns out that the vertex is located at coordinates $V(\,h, k\,)$.

Example 10.1. Find the vertex of the parabola $y = x^2 - 2x + 4$.

The previous example was surely of **Skill Level 0**; the completion of the square was no problem. In the next example, the completion process is slightly trickier.

Example 10.2. Find the vertex of the parabola $4x^2 + 3x + 2y = 3$.

Before asking you to do a few, let's summarize.

Finding the Vertex:
Given a second degree equation that has been put into the form $y = ax^2 + bx + c$, $a \neq 0$, the vertex can be obtained by completing the square. This having been done, put the equation into **standard form**:

$$y - k = a(x - h)^2 \tag{7}$$

The vertex is located at $V(\,h, k\,)$.

Now, let's go to the exercises. The solutions are on separate pages. You can look at them in order without seeing the solution to the next one. Use this to refine you solution methods.

Exercise 10.1. (Skill Level 0) Use the technique of completing the square to put each of the parabolas into standard form. Having done that, write the coordinates of the vertex in the form $V(\,h, k\,)$ and state whether the parabola opens up or down.
(a) $y = x^2 - 6x$ (b) $y = 1 - 4x - x^2$ (c) $x^2 + 2x + y + 1 = 0$

In the next exercise the problem of completing the square is a little trickier, but I expect you will get a score of 100%.

Exercise 10.2. (Skill Level 0.5) Use the technique of completing the square to put each of the parabolas into standard form. Having done that, write the coordinates of the vertex in the form $V(\,h, k\,)$ and state whether the parabola opens up or down.
(a) $y = 4x^2 + 2x + 1$ (b) $2x^2 - 3x - 2y + 1 = 0$
(c) $x - y = 4x^2$ (d) $3x^2 + 2x + 4y - 2 = 0$

We now turn to the topic of graphing.

▷ **Graphing.**

Graphing a parabola is very simple once you have obtained some basic information about the parabola:

- *Orientation.* Does the parabola open up/down, or right/left. This information can be easily observed using the classification rules.
- *Location.* The vertex determines the location of the parabola. See the discussion on finding the vertex.

Other points of interest in graphing a parabola are

- *Symmetry.* A parabola that is vertically oriented—it opens up or down—is symmetrical with respect to the vertical line passing through the vertex. This makes it even easier to graph.
- *Points to Plot.* It is not necessary to plot a large number of points to make a *rough sketch.* Usually, plotting the vertex, and two other points to determine the "breath" of the parabola are

sufficient. If the parabola crosses the x-axis, then fixing these point(s) is useful too.

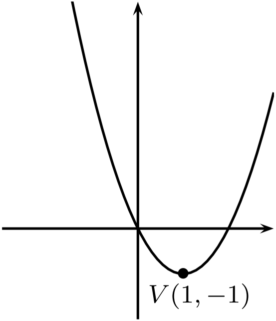


FIGURE 2(a) $y + 1 = (x - 1)^2$

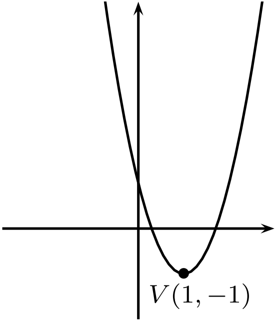


FIGURE 2(b) $y + 1 = 2(x - 1)^2$

These two parabolas have virtually the same equation; the only difference is in FIGURE 2(b), the $(x - 1)^2$ is multiplied by 2. This "scaling factor" tends to narrow the parabola. The get a feel for the breadth of the parabola, simply plot two points symmetrically placed on either side of the vertex. The pass a parabolic curve through these three points. **Three points and you're done!**

EXERCISE 10.3. Find the x-intercepts for the two parabolas in FIGURE 2: (a) $y+1=(x-1)^2$ (b) $y+1=2(x-1)^2$.

EXERCISE 10.4. Make a rough sketch of each of the following parabolas on the same sheet of paper.
(a) $y=x^2+1$ (b) $y=2x^2+1$ (c) $y=\frac{1}{2}x^2+1$

EXERCISE 10.5. Make rough sketches of the graphs of each of the following by putting each into standard form; classifying each as opening up or down; and finding the vertex of each.
(a) $4x+y-x^2=1$ (b) $y=-x^2-6x+1$ (c) $2x-3y=x^2$

- **y is of second degree**

When you have a second degree equation, like equation (2), and y has degree *two* with x only degree *one* ($A=0$ and $C\neq 0$), you have a parabola that opens either to the **left** or to the **right**. Here are a few visuals.

$$x=y^2 \quad x+2y=y^2 \quad 2x+3y+4y^2=1.$$

As in the previous case, to analyze these equations the basic skills we must master the basic skills: **recognition**, **classifcation**, **location**, and **graphing**.

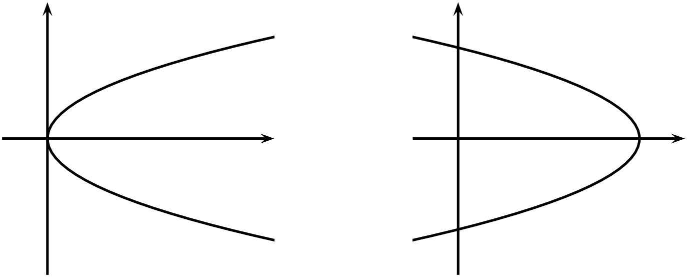


FIGURE 3(a) $x = y^2$
Parabola Opening Right

FIGURE 3(b) $x = 4 - y^2$
Parabola Opening Left

Notice the vertex is the point that is the *right-most* or *left-most* point on the graph, depending on the orientation of the parabola.

Because of the similarity of these case with the previous (extensively discussed) case, only exercises will be presented. In each exercise, observe that the parabolas are of the advertised type and that they equation define x as a function of y.

The general form of a parabola that opens to the left or right, written as a function of y, is

$$x = ay^2 + by + c$$

Here is all the relevant information.

Analyzing Horizontally Oriented Parabolas:
Consider the parabola

$$x = ay^2 + by + c, \quad a \neq 0. \tag{8}$$

The parabola **opens right** if $a > 0$ and **opens left** if $a < 0$; This equation can be put into the **standard form**

$$x - h = a(y - k)^2 \tag{9}$$

using the method of *completing the square*. The location of the vertex is $V(\, h, k\,)$.

EXERCISE 10.6. Analyze the following parabolas by putting each equation into **standard form**, equation (9), observing its *orientation*, finding its *vertex*, and, finally, making a rough sketch of the parabola. (The solutions to each are given on separate pages so you can, for example, look at the solution to (a) to refine your techniques to solve (b)–(d).)

(a) $x + 2y = y^2$ (b) $2x - 4y + y^2 = 3$
(c) $2y^2 - 4y + x = 0$ (d) $2x + 3y + 4y^2 = 1$

• Intersecting Curves

Quiz. Let $y = mx + b$ and $y = ax^2 + bx + c$ be a line and a parabola, respectively. What is the largest number of points of intersection possible between these two curves?

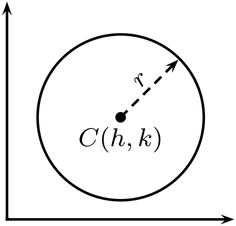
(a) 0 (b) 1 (c) 2 (d) 3

Let's have a few "reminder" exercises in the form of solving equations. Try to graphically visualize the questions; the graphing fanatic might even graph all these equations . . . perhaps using a graphing calculator.

Exercise 10.7. Find the coordinates of intersection between the two given curves. (*Recall*: The Quadratic Formula.)

(a) Write out the general strategy of solving these problems.
(b) $y = 2x + 1$; $y = x^2 + 2x - 1$ (c) $y = 2x + 1$; $y = x^2 + 4x - 1$
(d) $y = x^2 + 2$; $y = 2x^2 - 3x + 3$ (e) $y = x^2 + 2$; $y = 4 - (x - 1)^2$

10.2. Circles



Let $r > 0$ be a number greater than zero and $C(h, k)$ a point in the plane. The **circle** with **center** at $C(h, k)$ and **radius** r is the set of all points $P(x, y)$ that are a distance of r units from the point C. In terms of symbols, a point P is on this circle if (and only if) it satisfies the equation

$$d(P, C) = r, \tag{10}$$

where $d(P, C)$ is the distance between the points P and C.

Now, invoke the distance formula with $P(x, y)$ and $C(h, k)$ to obtain

$$\sqrt{(x-h)^2 + (y-k)^2} = r \tag{11}$$

Square both sides to obtain the **center-radius form** for the equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Summary:

Center-Radius Form of a Circle
The equation of the circle having radius $r > 0$ and center at $C(h, k)$ is given by

$$(x - h)^2 + (y - k)^2 = r^2 \tag{12}$$

Illustration 2. Here are a few simple examples of circles already in the center-radius form; in this case, basic information can be extracted about the circle.

(a) $x^2 + y^2 = 1$ is a circle of radius $r = 1$ with center at $(0, 0)$.
(b) $(x - 1)^2 + (y - 3)^2 = 16$ is a circle with radius $r = 4$ ($r^2 = 16$) and center at $C(1, 3)$.
(c) $(x + 2)^2 + (y - 4)^2 = 5$ is a circle with radius $r = \sqrt{5}$ ($r^2 = 5$) and center at $C(-2, 4)$. ■

To construct the equation of a circle, therefore, you need two pieces of information: (1) the center of the circle and (2) the radius of the circle. All your efforts must be concentrated towards acquiring these two.

Exercise 10.8. Find the equation of the circle with center at C and passes through the given point P

(a) Write out a strategy for solving this type of problem.

(b) $C(1, 2)$; $P(4, -1)$ (c) $C(-2, 3)$; $P(5, 2)$

Exercise 10.9. Find the equation of the circle that passes through the two diametrically opposite points P_1 and P_1.

(a) Write out a strategy for solving this type of problem.

(b) $P_1(1, 3)$ and $P_2(3, 5)$ (c) $P_1(-3, 1)$ and $P_2(4, 6)$

Normally, the circle is initially *not* in the center-radius form, but is often in the **general form**. If you expand (multiply out) equation (12) and write it in the form of a second degree equation, equation (2), you get

$$\boxed{Ax^2 + Ay^2 + Bx + Cy + D = 0, \quad A \neq 0} \tag{13}$$

The equation is called the **General Form** for the equation of a circle.

Important Points. Notice the coefficients of the two squared terms are equal (to A). This (almost) characterizes a second degree equation as being a circle. It is also important to emphasize that the coefficients of the squared terms have the same *sign*! For example, $2x^2 + 2y^2 + x + y - 1 = 0$ is a circle but $2x^2 - 2y^2 + x + y - 1 = 0$ is *not*! ■

Here are a few quick visual manifestations of equation (13).

Illustration 3. Examples of Circles in General Form.

(a) $x^2 + y^2 + 2x - 3y + 1 = 0$.
(b) $2x^2 + 2y^2 - 4x + 8y + 3 = 0$
(c) $-3x^2 - 3y^2 + 3x - 4y + 2 = 0$. Here the common coefficient of the squared terms is $A = -3$, that's o.k. Needless to say, this equation can be rewritten in a more esthetically pleasing form: $3x^2 + 3y^2 - 3x + 4y - 2 = 0$. ■

▷ **Transforming to the Center-Radius Form.** The center-radius form is an "information yielding form" for a circle; therefore it is beneficial to put an equation representing a circle into this form.

The Method of Transforming. Given an equation on the form of equation (13), you can complete the square of the x-terms and complete the square of the y-terms to obtain the **center-radius form** of the equation of a circle.

EXAMPLE 10.3. Put each of the following circles in the center-radius form and find their center and radius.
(a) $x^2 + y^2 - 2x + 4y + 1 = 0$ (b) $2x^2 + 2y^2 + 4x - y - 2 = 0$

You see from the example the skills of needed to put an equation in the center-radius form are skills you have been practicing in the past several lessons. The other important skill is *interpretation of results*! It doesn't do you much good if you can't extract the information (center and radius).

I should mention, before we go to the exercises that not every equation of the form (13) is a circle. Here are three very simple examples to illustrate the possibilities:

(1) $x^2 + y^2 = 1$ ◁ It's graph is a **circle**

(2) $x^2 + y^2 = 0$ ◁ It's graph is a **point**

(3) $x^2 + y^2 = -1$ ◁ This equation has **no graph!**

In (2), the only point (x, y) that satisfies the equation is $(0, 0)$; hence, its graph consists of a single point. (A degenerate circle?) In (3), there is no point (x, y) that satisfies the equation, so there are no points on the graph.

EXERCISE 10.10. Put each of the following into the center-radius form. If an equation represents a circle, state its center and radius and if it does not represent a circle characterize its graph.

(a) $x^2 + y^2 - 6x + 2y - 4 = 0$ (b) $x^2 + y^2 + 4y = 2$

(c) $3x^2 + 3y^2 - 8x = 0$ (d) $x^2 + y^2 + 2y - 4x + 10 = 0$

Exercise 10.11. Find the points of intersection between the two given curves.

(a) $x^2 + y^2 = 4$; $y = 2x$ (b) $x^2 + y^2 = 4$; $y = x + 1$
(c) $x^2 + y^2 - 2x = 4$; $y = 2x - 1$ (d) $x^2 + y^2 - 2x = 4$; $x + 2y = 6$.

10.3. The Trig Functions

The trigonometric functions play an important role in many scientific and technical fields as well as in many of the trades (carpentry comes to mind). In this section we give a less than complete presentation of some basic ideas.

• The Definitions and Consequences

The definitions of the trig functions are quite simple. For any number t, we want to define $\cos(t)$ and $\sin(t)$. This is done using the unit circle, $x^2 + y^2 = 1$.

The Wrapping Definition of Sine and Cosine: Let t be any given real number. Draw the unit circle $x^2 + y^2 = 1$. Beginning at the point $I(1, 0)$, called the *initial point*, measure off a length of t units around the unit circle; if $t \geq 0$, mark off this length *counter clockwise* and if $t < 0$, mark off $|t|$ units in a *clockwise direction*. The point on the circle at the end of your measurement process is called the *terminal point*, (temporarily) denoted by $P_t(x, y)$.

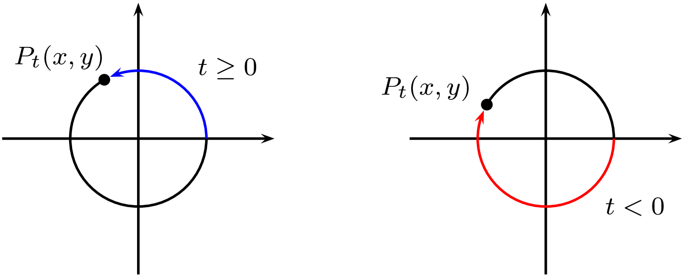


FIGURE 4

If the terminal point is given by $P_t(x, y)$, then define

$$\cos(t) = x \quad \text{and} \quad \sin(t) = y \tag{14}$$

That is, the $\cos(t)$ is defined to be the first coordinate of the terminal point, and the $\sin(t)$ is defined as the second coordinate.

Let's list off some simple consequences of this wrapping definition.

▷ *The Fundamental Identity for Sine and Cosine*:

$$\boxed{\cos^2(t) + \sin^2(t) = 1} \tag{15}$$

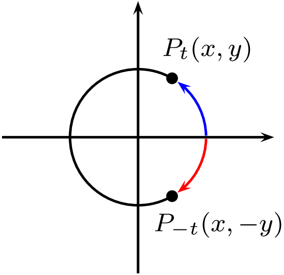
This is because, by our definition, the $\cos(t)$ and $\sin(t)$ are the first and second coordinates, respectively, of the terminal point $P_t(x, y)$. But P_t is a point on the unit circle and as such, its coordinates satisfy the equation $x^2 + y^2 = 1$; therefore, $\cos^2(t) + \sin^2(t) = 1$ *for any number* t. ■

▷ *Symmetry Properties of Sine and Cosine*:

$$\boxed{\cos(-t) = \cos(t) \qquad \sin(-t) = -\sin(t)} \tag{16}$$

We say a function $f(t)$ is an *even function* if it satisfies the equation $f(-t) = f(t)$ for all t in its domain; consequently, we may say that $\cos(t)$ is an *even function*. A function $f(t)$ is called an *odd function* if it satisfies the equation $f(-t) = -f(t)$ for all t in its domain; therefore, we are entitled to say that $\sin(t)$ is an *odd function*.

The proof of equation (16) is given using the wrapping definition of sine and cosine.



Think of t as positive. Measure t units counter clockwise starting at $I(0, 1)$ and terminating at $P_t(x, y)$. (This is shown in blue). By definition, $\cos(t) = x$ and $\sin(t) = y$. Now take the *negative* number $-t$ and measure off t units clockwise around the circle, shown in red, to the terminal point $P_{-t}(x, -y)$. Because we are wrapping the same length, but in opposite directions, the x-coordinate of P_{-t} will be the same as the x-coordinate of P_t; the y-coordinate of P_{-t} will be of *negative* the y-coordinate of P_t. Since the terminal point of $-t$ is $P_{-t}(x, -y)$, by definition we have that $\cos(-t) = x$ and $\sin(-t) = -y$; but $x = \cos(t)$ and $y = \sin(t)$. Putting these observations together we get

$$\cos(-t) = x = \cos(x) \qquad \sin(-t) = -y = -\sin(t),$$

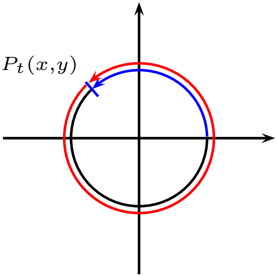
which are the symmetry properties.

▷ *Periodicity of Sine and Cosine*:

$$\boxed{\cos(t+2\pi) = \cos(t) \qquad \sin(t+2\pi) = \sin(t)} \tag{17}$$

In this case we say that $\cos(t)$ and $\sin(t)$ are *periodic* of *period* 2π. (In general, a function $f(t)$ is periodic of period p provided $f(t+p) = f(t)$ for all t in its domain. Not all functions are periodic.)

Again, this property is best seen through a picture.



For any given number t, assumed to be positive for the purposes of demonstration, wrap t units around the unit circle, as shown in blue. Denote the terminal point by $P_t(\cos(t), \sin(t))$. Now compute the cosine and sine of the number $t+2\pi$. Begin by measuring off $t+2\pi$ units around the circle; we do this by first measuring off t units (shown in blue) followed by an *additional* 2π units (shown in red). Recalling that the circumference of the unit circle is 2π, we see that when we wrap the additional 2π around, we are just going in circles! :-) Therefore, P_t is the same as $P_{t+2\pi}$; in words, the terminal point for the number t is the same as the terminal point for the number $t + 2\pi$:

$$P_t(x, y) = P_{t+2\pi}(x, y).$$

It follows, from the wrapping definition, that

$$\cos(t) = x = \cos(t + 2\pi) \qquad \sin(t) = y = \sin(t + 2\pi),$$

which are the periodic equations.

▷ *The Range of Cosine and Sine*: We first observe that all coordinates on the unit circle are between -1 and 1. Because the $\cos(t)$ and $\sin(t)$ are coordinates on the unit circle, it follows

$$\boxed{-1 \le \cos(t) \le 1 \qquad -1 \le \sin(t) \le 1} \tag{18}$$

• Some Common Values

In the age of the hand-held graphing calculator, why, you may ask, should I know some of the common values of the sine and cosine? The short answer is that some instructors, including yours truly, expect the student to know these values. These values are among the "minimal set of knowledge" anyone talking to you about the trig functions would expect you to know. You want to be at least *minimal*, don't you? :-)

The circumference of the unit circle is 2π. The numbers $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$, when "wrapped" around the unit circle, go zero of the way around, a quarter of the way around, a half of the way around, three-quarters the way around, and all the way around, respectively. The terminal points of these five values of t can be easily observed:

$$P_0(1,0) \quad P_{\pi/2}(0,1) \quad P_\pi(-1,0) \quad P_{3\pi/2}(0,-1) \quad P_{2\pi}(1,0) \tag{19}$$

The coordinates of these "easy" points contain the cosines and sines of the corresponding values of t. (Given in the subscript of the point, I might add.)

• Quiz. As a test of your understanding of the wrapping definition, and the meaning of the above notation, take the following quiz on some of the common values of sine and cosine by clicking on the green bullet. ∎

Now let's concentrate on three other values of t: $t = \pi/6$, $t = \pi/4$, and $t = \pi/3$. These can be worked out rather easily, though it will not be done in these lessons:

$$P_{\pi/6}(\sqrt{3}/2, 1/2) \quad P_{\pi/4}(\sqrt{2}/2, \sqrt{2}/2) \quad P_{\pi/3}(1/2, \sqrt{3}/2)$$

Look at the corresponding unit circle with these values wrapped.

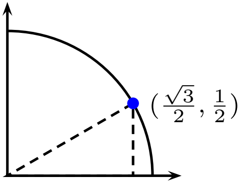


FIGURE 5(a) $t = \frac{\pi}{6}$

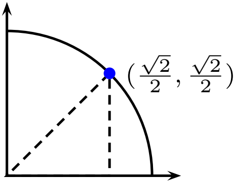


FIGURE 5(b) $t = \frac{\pi}{4}$

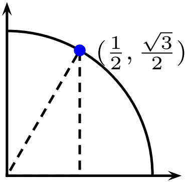


FIGURE 5(c) $t = \frac{\pi}{3}$

In FIGURE 5(a), we see that the $\cos(\pi/6) = \sqrt{3}/2$ and $\sin(\pi/6) = 1/2$. These values correspond to the base and height of the right triangle. (You can see that the height of the triangle is (about) 1/2 the radius of the unit circle.) The companion values, shown in FIGURE 5(c), are $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$. (Again notice the base is of the triangle is about 1/2 the radius.) Finally, FIGURE 5(c) tells us that $\cos(\pi/4) = \sqrt{2}/2$ and $\sin(\pi/4) = \sqrt{2}/2$.

Here is a table and figure summarizing the values of sine and cosine.

t	$\cos(t)$	$\sin(t)$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1

TABLE 1

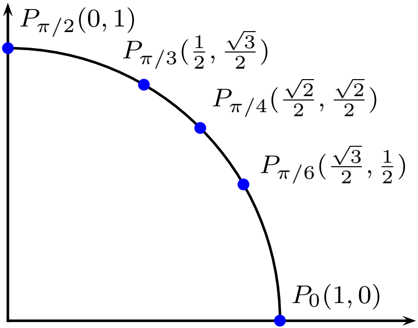


FIGURE 6

By utilizing the symmetry property, the periodic property, and the wrapping definition of sines and cosines you can *easily* calculate any integer multiple of $\frac{\pi}{n}$, where $n = 1, 2, 3, 4, 6$. In particular, it is important to keep a visualization of the wrapping pictures of FIGURE 4.

In the example that follows, I exhibit the simple reasoning necessary to calculate any value integer multiple of $\dfrac{\pi}{n}$, for $n = 1, 2, 3, 4, 6$.

EXAMPLE 10.4. Calculate the *exact value* of each of the following:
(a) $\sin(-\pi/3)$ (b) $\cos(2\pi/3)$ (c) $\cos(5\pi/4)$ (d) $\sin(-\pi/2)$

Tip. *How to Determine Values of Sine and Cosine*: To calculate the sine or cosine of multiples of π/n ($n = 1, 2, 3, 4, 6$) (1) draw a unit circle; (2) plot the terminal point (this may require division, e.g.,

$$\frac{13\pi}{6} = \frac{13}{6}\pi = 2\tfrac{1}{6}\,\pi = 2\pi + \frac{\pi}{6},$$

thus, $13\pi/6$ is $\pi/6$ "beyond" 2π); (3) observe the relation between this point and one of the terminal points given in TABLE 1; (4) calculate value with the proper sign affixed. ■

Note. It is actually only necessary to wrap *clockwise*; i.e., you need only deal with positive values of t. This is because of the symmetry properties; thus,

$$\sin(-7\pi/3) = -\sin(7\pi/3) \quad \text{need only find } \sin(7\pi/3)$$
$$\cos(-9\pi/4) = \cos(9\pi/4) \quad \text{need only find } \cos(9\pi/4)$$

EXERCISE 10.12. Find the values of each of the following *without the aid of a calculator*.
(a) $\sin(3\pi/4)$ (b) $\cos(7\pi/6)$ (c) $\sin(-7\pi/6)$ (d) $\cos(-21\pi/2)$
(e) $\sin(9\pi/3)$ (f) $\cos(10\pi/3)$ (g) $\sin(10\pi/3)$ (h) $\cos(-29\pi/6)$

• Graphs of the Trig Functions

Having defined, for any number t, the $\cos(t)$ and $\sin(t)$, we now want to study them as functions. Usually, the independent variable of a function is x, not t. Replace the letter t by the letter x to obtain two trigonometric *functions*:

$$y = \cos(x) \qquad y = \sin(x)$$

In this brief section, we remind the reader of the graphs of these two functions.

▷ *The Graph of* $y = \cos(x)$. Let us begin graphing the cosine and making some comments.

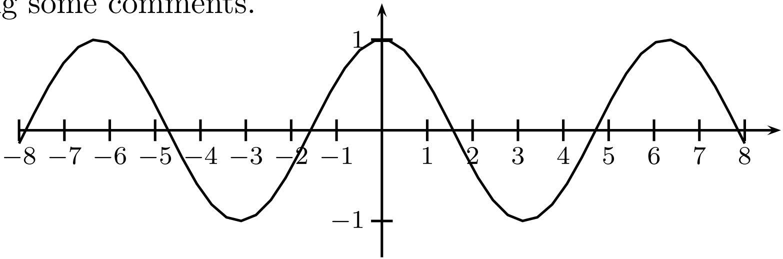


FIGURE 7 $y = \cos(x)$

The fact that the graph is *symmetric with respect* to the y-axis is a manifestation of the even function property of the cosine. The regular, repeated behavior of the graph is the periodicity of the cosine. The cosine function repeats its pattern of values over consecutive intervals of length 2π, the period.

The x-intercepts of the cosine are all values of x for which $\cos(x) = 0$. Based on the wrapping definition of the cosine, it is easy to see that the x-intercepts are

$$x = \ldots, \; -\frac{5\pi}{2}, \; -\frac{3\pi}{2}, \; -\frac{\pi}{2}, \; \frac{\pi}{2}, \; \frac{3\pi}{2}, \; \frac{5\pi}{2}, \ldots \tag{20}$$

Or, in other words, the graph of $y = \cos(x)$ crosses the x-axis at all *odd multiples* of $\pi/2$. This is an important point.

The $\cos(x)$ attains its maximum value of $y = 1$ at

$$x = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots \tag{21}$$

In other words, $\cos(x)$ takes on its maximum value at all *even multiples of* π.

EXERCISE 10.13. List, in a style similar to equation (21), the values of x at which the $\cos(x)$ takes on its minimum value of $y = -1$. Having done that, write a good English sentence that describes these numbers. (See the sentence that follows equation (21) above.)

▷ *The Graph of* $y = \sin(x)$. Let us begin graphing the sine and making some comments.

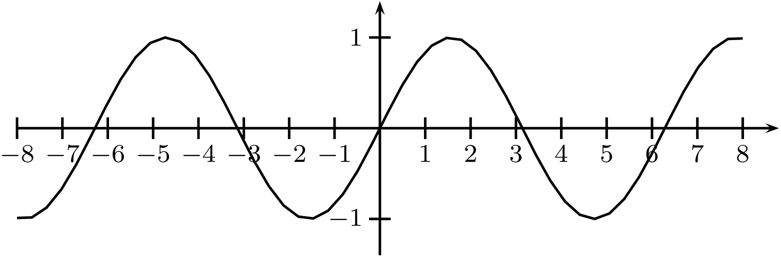


FIGURE 8 $y = \sin(x)$

The sine function is an odd function; the graph of an odd function is *symmetrical* with respect to the *origin*. The regular repeated behavior is the *periodicity* of the sine function. The sine function repeats its pattern of values over consecutive intervals of length 2π, the period.

EXERCISE 10.14. **The x-intercepts.** List out the x-intercepts of $y = \sin(x)$ using the style of equation (21), then describe this set of points using a good English sentence. (*References*: TABLE 1 and FIGURE 8.)

EXERCISE 10.15. **Maximum Values of** $\sin(x)$**.** List the values of x at which the function $y = \sin(x)$ attains its maximum value of $y = 1$. Then describe these numbers in a good English sentence.

EXERCISE 10.16. **Minimum Values of** $\sin(x)$**.** List the values of x at which the function $y = \sin(x)$ attains its minimum value of $y = -1$. Then describe these numbers in a good English sentence.

▷ *The Two Graphed Together*: It may be useful to view the graphs of the sine and cosine on one sheet of electronic paper in order to make a direct comparison between the two.

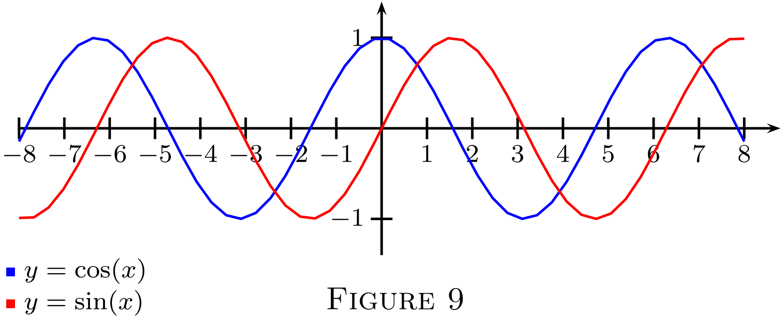


FIGURE 9

Notice that the graphs of $y = \cos(x)$ and $y = \sin(x)$ *appear* to be identical; identical in the sense that one graph, say $y = \cos(x)$, can be obtained by *shifting* the graph of $y = \sin(x)$ to the *left*. This observation is in fact correct; more precisely. *The cosine graph can be obtained from the sine graph by shifting the sine graph over* $\frac{\pi}{2}$ *units to the left.*

• The Other Trig Functions

There are four other trig functions that hitherto I have failed to mention. Their definitions are in terms of $\cos(x)$ and $\sin(x)$. These four are the tangent, cotangent, secant, and cosecant functions:

$$\begin{array}{ll} \tan(x) = \dfrac{\sin(x)}{\cos(x)} & \sec(x) = \dfrac{1}{\cos(x)} \\[2ex] \cot(x) = \dfrac{\cos(x)}{\sin(x)} & \csc(x) = \dfrac{1}{\sin(x)} \end{array} \tag{22}$$

Of course, we make these definitions for all values of x for which the denominators are nonzero.

The trigonometric functions play diverse roles in mathematics.

- They are quite commonly used to develop relationships between the sides of right triangles. These relationships are important in applications in such different fields as carpentry, architecture, surveying, navigation, and computer graphics ... to name just a few.
- In **Calculus**, these functions are studied like any other function. The trigonometric functions enable us to solve a greater variety of problems that require the special techniques of calculus. Having trigonometric functions enables scientist and engineers to properly formulate and solve complex problems, often with the aid of calculus and geometry.

The interesting and very useful applications must be reserved for a proper course in algebra, trigonometry, calculus, and/or courses in specialized fields. These lessons are meant as a *minimal review* of bygone memories.

Example 10.5. Identify the domain of definition of the function $y = \tan(x)$.

Quick Quiz. Is the domain of $\sec(x)$ the same as that of $\tan(x)$?

(a) Yes (b) No

EXERCISE 10.17. Identify the domain of definition of the function $y = \cot(x)$. (*Hint*: The domain would be all values of x for which the denominator is nonzero.)

- **Radian Measure versus Degree Measure**

The wrapping definition of sine and cosine tells us how to compute the sine and cosine of a number t. The number t is *not* interpreted as an angle, but was simply any real number. Within the context of the wrapping definition, the number t is said to be measured in *radians*.

What is angle measurement? As you wrap the number t around the unit circle, you are revolving around the center of the circle The distance around the circle is its circumference 2π; one complete revolution around the circle is referred to as a $360°$ revolution.

In this case we say 2π radians is the same as $360°$ (degrees). The degree measurement of an arbitrary number t is calculated by direct proportions; let the symbol θ denote the degree measurement of t, then using direct proportions:

$$\frac{\theta}{t} = \frac{360^\circ}{2\pi}$$

thus,

$$\theta = t \cdot \frac{360^\circ}{2\pi} = t \cdot \frac{180^\circ}{\pi}$$

The number $180/\pi$ is referred to as the *scaling factor* for converting *radians* to degrees.

Converting Radians to Degrees:
Let t be a number (measured in radians). The degree measurement of t is given by

$$\theta = t \cdot \frac{180}{\pi} \tag{23}$$

Illustration 4. Calculate the degree measurement of each of the following. by applying the conversion formula, equation (23).

(a) For $t = \dfrac{\pi}{4}$, $\theta = \dfrac{\pi}{4}\dfrac{180^\circ}{\pi} = \dfrac{180^\circ}{4} = \boxed{45^\circ}$.

(b) For $t = -\dfrac{\pi}{6}$, $\theta = -\dfrac{\pi}{6}\dfrac{180^\circ}{\pi} = -\dfrac{180^\circ}{6} = \boxed{-30^\circ}$. ■

This is not hard so I'll just ask you to make a few calculations yourself.

Exercise 10.18. Calculate the degree measurement of each of the following. by applying the conversion formula, equation (23).

(a) $\dfrac{2\pi}{3}$ (b) $-\dfrac{5\pi}{6}$ (c) 3π (d) $-\dfrac{3\pi}{4}$ (e) $\dfrac{5\pi}{2}$

In the same way we can convert degrees to radians by simply solving the equation (23) for t:

$$t = \theta \cdot \frac{\pi}{180} \implies t = \theta \cdot \frac{\pi}{180}.$$

Elevating this to the status of shadow box we obtain the following.

Converting Radians to Degrees:
Let θ be a number measured in degrees. The radian measurement of θ is given by

$$t = \theta \cdot \frac{\pi}{180} \tag{24}$$

Applying this formula is the same as applying the conversion formula (23) so no examples will be presented.

EXERCISE 10.19. Convert degrees to radians in each of the following.
(a) $30°$ (b) $60°$ (c) $45°$ (d) $-300°$ (e) $225°$

Naturally, if θ is a number measured in degree, we naturally define $\cos(\theta) = \cos(t)$ and $\sin(\theta) = \sin(t)$, where t is the radian measurement of θ, as computed by equation (24). For example,

$$\cos(30°) = \cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

Let's revisit TABLE 1 and include degree measurement. Below is a table of common values of sine and cosine that you should and must know.

t	θ	$\cos(t)$	$\sin(t)$
0	0°	1	0
$\pi/6$	30°	$\sqrt{3}/2$	$1/2$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	60°	$1/2$	$\sqrt{3}/2$
$\pi/2$	90°	0	1

TABLE 2

The angle measurement is usually seen in the context of problems involving triangles. These problem types will not be covered in these lessons—sorry.

As a final set of thoughts, let's connect up the geometric relationship between radian measure and degree measure.

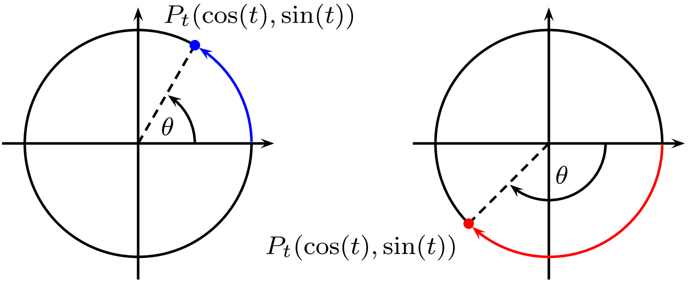


FIGURE 10

The degree measure is a measure of how many times we wrap around the unit circle based on 360° rather than 2π, as is the case of radian measure. The student needs to be able to move quickly and quietly between these two systems of measurement.

This point represents the end of our ALGEBRA REVIEW IN TEN LESSONS. It ended not with a bang but a measure!

If you have waded through all ten lessons, my hearty congratulations to you. There was much of algebra and trigonometry not covered in these lessons. These electronic tomes were not meant to be an entire course, but a review of basic concepts and techniques.

In additional to the basic techniques and ideas presented, I hope you come away from these lessons with the ability to use *good and standard notation* and to present your thoughts in a clear and concise manner. This would be most appreciated by your professors, and is a sign of your mathematical maturity.

I hope these imperfect lessons have helped you review some of the basics concepts your professors will assume you know. Now, on to e-Calculus!

To exit, either go to the top of this file and follow the arrows, or simply click here. DPS

Solutions to Exercises

10.1. *Solution*: We simple complete the square!

Solution to (a). Analyze: $y = x^2 - 6x$.

$$\begin{aligned} y &= x^2 - 6x && \triangleleft \text{ given} \\ y &= (x^2 - 6x + 9) - 9 && \triangleleft \text{ take } \tfrac{1}{2} \text{ coeff. of } x \\ y &= (x - 3)^2 - 9 && \triangleleft \text{ the square is completed!} \end{aligned}$$

Now we put this last equation in standard form:

$$\boxed{y + 9 = (x - 3)^2}$$

We see this is a parabola that *opens down* and has its vertex at $\boxed{V(3, -9).}$

Solution to (b) Analyze: $y = 1 - 4x - x^2$.

$$y = 1 - 4x - x^2 \quad \triangleleft \text{ given. Now assoc. } x \text{ terms}$$

$$y = 1 + (-4x - x^2) \quad \triangleleft \text{ Now, factor our coeff. of } x^2$$

$$y = 1 - (x^2 + 4x) \quad \triangleleft \text{ Next, take } \tfrac{1}{2} \text{ coeff. of } x$$

$$y = 1 - (x^2 + 4x + 4) + 4 \quad \triangleleft \begin{cases} \text{Adding 4 inside parentheses} \\ \text{same as adding 4 outside} \end{cases}$$

$$y = 5 - (x + 2)^2 \quad \triangleleft \text{ basically done!}$$

Now write the last equation in standard form:

$$\boxed{y - 5 = -(x + 2)^2}$$

This is a parabola that *opens down* and has its vertex at $\boxed{V(-2, 5).}$

Solution to (c) Analyze: $x^2 + 2x + y + 1 = 0$.

Begin by writing the equation as a function of x;

$$y = -1 - x^2 - 2x$$

and not continue as before:

$$
\begin{aligned}
y &= -1 - x^2 - 2x && \triangleleft \text{ given} \\
y &= -1 - (x^2 + 2x) && \triangleleft \text{ factor our coeff. of } x^2 \\
y &= -1 - (x^2 + 2x + 1) + 1 && \triangleleft \begin{cases} \text{Adding 1 inside parentheses} \\ \text{same as adding 1 outside} \end{cases} \\
y &= -(x + 1)^2 && \triangleleft \text{ done!}
\end{aligned}
$$

Summary: This is a parabola that *opens down* having vertex located at $\boxed{V(-1, 0).}$ (Here, relative to the notation of the standard form, $k = 0$.)

Exercise 10.1. ■

10.2. *Solution to* (a) Analyze: $y = 4x^2 + 2x + 1$.

$$
\begin{aligned}
y &= 4x^2 + 2x + 1 && \triangleleft \text{ given} \\
y &= 4(x^2 + \tfrac{1}{2}x) + 1 && \triangleleft \text{ factor our coeff. of } x^2 \\
y &= 4(x^2 + \tfrac{1}{2}x + \tfrac{1}{16}) + 1 - \tfrac{1}{4} && \triangleleft \text{ complete square} \\
y &= 4(x + \tfrac{1}{4})^2 + \tfrac{3}{4} && \triangleleft \text{ write as perfect square}
\end{aligned}
$$

Summary: The equation in standard form is

$$\boxed{y - \tfrac{3}{4} = 4(x + \tfrac{1}{4})^2}$$

This is a parabola that opens up and has vertex at $\boxed{V(\,-\tfrac{1}{4}, \tfrac{3}{4}\,).}$

Solution to (b) Analyze: $2x^2 - 3x - 2y + 1 = 0$. Begin by solving for y as a function of x:

$$y = x^2 - \tfrac{3}{2}x + \tfrac{1}{2}$$

and proceed using standard techniques.

$$y = (x^2 - \tfrac{3}{2}x) + \tfrac{1}{2}$$
$$y = (x^2 - \tfrac{3}{2}x + \tfrac{9}{16}) + \tfrac{1}{2} - \tfrac{9}{16} \quad \triangleleft \text{ complete square}$$
$$y = (x - \tfrac{3}{4})^2 - \tfrac{1}{16} \quad \triangleleft \text{ write as perfect square}$$

Summary: The equation written standard form is

$$\boxed{y + \tfrac{1}{16} = (x - \tfrac{3}{4})^2.}$$

This equation describes a parabola that opens up and has its vertex at $\boxed{V(\,\tfrac{3}{4}, -\tfrac{1}{16}\,).}$

Solution to (c) Analyze: $x - y = 4x^2$. Begin by solving for y as a function of x:

$$y = -4x^2 + x$$

and complete the square.

$$y = -4(x^2 - \tfrac{1}{4}x) \quad \triangleleft \text{ factor out leading coeff.}$$
$$y = -4(x^2 - \tfrac{1}{4}x + \tfrac{1}{64}) + \tfrac{1}{16} \quad \triangleleft \text{ complete square}$$
$$y = -4(x - \tfrac{1}{8})^2 + \tfrac{1}{16} \quad \triangleleft \text{ write as perfect square}$$

This last equation written is standard form is

$$\boxed{y - \tfrac{1}{16} = -4(x - \tfrac{1}{8})^2.}$$

This is a parabola that opens down with vertex as $\boxed{V(\,\tfrac{1}{8}, \tfrac{1}{16}\,).}$

Solution to (d) Analyze: $3x^2 + 2x + 4y - 2 = 0$. Write as a function of x:

$$y = -\tfrac{3}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{2}$$

and complete the square.

$y = -\frac{3}{4}x^2 - \frac{1}{2}x + \frac{1}{2}$	◁ given
$y = -\frac{3}{4}(x^2 + \frac{2}{3}x) + \frac{1}{2}$	◁ factor out leading coeff.
$y = -\frac{3}{4}(x^2 + \frac{2}{3}x + \frac{1}{9}) + \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{9}$	◁ complete square
$y = -\frac{3}{4}(x + \frac{1}{3})^2 + \frac{7}{12}$	◁ write as perfect square

Summary: Write in standard form:

$$\boxed{y - \tfrac{7}{12} = -\tfrac{3}{4}(x + \tfrac{1}{3})^2}$$

This equation represents a parabola that opens down and has vertex at $\boxed{V(-\frac{1}{3}, \frac{7}{12})}$.

Remarks: As you can see, these are all solved exactly the same way. The only difference is the level of difficulty in completing the square.

This is why fundamental algebra is so important: It is used to extract information. Unless you can perform the algebra, you cannot extract the information—not good in the information age! Exercise 10.2. ■

10.3. *Solutions*

(a) Find the x-intercepts of $y + 1 = (x - 1)^2$. We use standard methods. To find the x-intercept, we put $y = 0$ and solve for x.

$$1 = (x-1)^2 \text{ or } (x-1)^2 = 1$$

We now solve for x:

$$\begin{aligned} |x - 1| &= 1 && \triangleleft \text{ take square root of both sides} \\ x - 1 &= \pm 1 && \triangleleft \text{ solve absolute equation} \\ x &= 1 \pm 1 && \triangleleft \text{ solve for } x \\ x &= 0,\ 2 && \triangleleft \text{ enumerate solutions} \end{aligned}$$

Presentation of Answer: The x-intercepts are $\boxed{x = 0,\ 2.}$

(b) Find the x-intercepts of $y + 1 = 2(x - 1)^2$. Put $y = 0$ and solve for x.

$$1 = 2(x - 1)^2 \text{ or } (x - 1)^2 = \tfrac{1}{2}$$

Now solve for x:

$$|x - 1| = \frac{1}{\sqrt{2}} \quad \triangleleft \text{ take square root of both sides}$$

$$x - 1 = \pm\frac{1}{\sqrt{2}} \quad \triangleleft \text{ get rid of absolute values}$$

$$x = 1 \pm \frac{1}{\sqrt{2}} \quad \triangleleft \text{ solve for } x$$

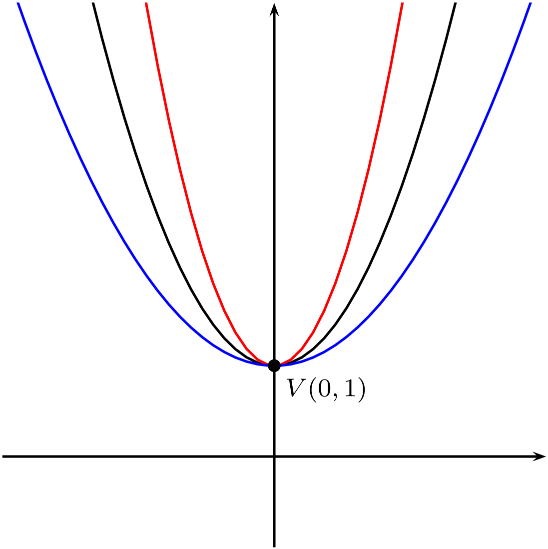
Presentation of Answer: The x-intercepts are at

$$\boxed{x = 1 - \frac{1}{\sqrt{2}},\ 1 + \frac{1}{\sqrt{2}}.}$$

Notice they are symmetrically placed on either side of the vertex.

Exercise 10.3. ■

10.4. *Solution*: Your rough graphs should roughly look like the following:



Legend: ■ $y = x^2 + 1$; ■ $y = 2x^2 + 1$; ■ $y = \frac{1}{2}x^2 + 1$.

Exercise 10.4. ■

10.5. *Solutions*: Here are outlines of the solutions.

(a) Analyze: $4x + y - x^2 = 1$.

$$\begin{aligned} 4x + y - x^2 &= 1 && \triangleleft \text{ given} \\ y &= x^2 - 4x + 1 && \triangleleft \text{ re-write} \\ y &= (x - 4x + 4) + 1 - 4 && \triangleleft \text{ complete square} \\ y + 3 &= (x - 2)^2 && \triangleleft \text{ write in proper form} \end{aligned}$$

Summary: This is a parabola that *opens up* with vertex at $V(\,2, -3\,)$, the sketch is left to the reader.

(b) Analyze: $y = -x^2 - 6x + 1$.

$$\begin{aligned} y &= -x^2 - 6x + 1 && \triangleleft \text{ given} \\ y &= -(x^2 + 6x) + 1 && \triangleleft \text{ factor out leading coeff.} \\ y &= -(x^2 + 6x + 9) + 1 + 9 && \triangleleft \text{ complete square} \\ y - 10 &= -(x + 3)^2 && \triangleleft \text{ write in proper form} \end{aligned}$$

Summary: This is a parabola that *opens down* and having vertex at $V(-3, 10)$.

(c) Analyze: $2x - 3y = x^2$.

$$\begin{aligned}
2x - 3y &= x^2 && \triangleleft \text{ given} \\
-3y &= x^2 - 2x && \triangleleft \text{ isolate} \\
-3y &= (x^2 - 2x + 1) - 1 && \triangleleft \text{ complete square} \\
-3y &= (x-1)^2 - 1 && \triangleleft \text{ write as perfect square} \\
y &= -\tfrac{1}{3}(x-1)^2 + \tfrac{1}{3} && \triangleleft \text{ divide through} \\
y - \tfrac{1}{3} &= -\tfrac{1}{3}(x-1)^2 && \triangleleft \text{ put in standard form}
\end{aligned}$$

Summary: This is the equation of a parabola that *opens down* with vertex at $V(1, \frac{1}{3})$.

Exercise Notes: In this last part, I did something a little different. I completed the square of the l.h.s. before dividing through by the coefficient of y; this was to avoid working with fractional expressions unnecessarily. Many of the previous examples and exercises could have been handled the same way. ∎

Exercise 10.5. ∎

10.6. *Solution to* (a) Analyze: $x + 2y = y^2$.

First write the equation as a function of y:

$$x = y^2 - 2y.$$

Now, complete the square

$$x = (y^2 - 2y + 1) - 1 \quad \triangleleft \text{ take } \tfrac{1}{2} \text{ coeff. of } y$$

$$x = (y - 1)^2 - 1 \quad \triangleleft \text{ write as perfect square}$$

Summary: The equation written in standard form is

$$\boxed{x + 1 = (y - 1)^2}$$

This is a horizontally oriented parabola (since y has degree 2 and x has degree 1), it opens to the right (since the coefficient of the y^2 term is positive), and its vertex is located at $V(, -1, 1)$.

The sketch of the graph is left to the student—that's you. 𝔇𝔓

Solution to (b) Analyze: $2x - 4y + y^2 = 3$.

Begin by solving for x; writing x as a function of y:

$$x = -\tfrac{1}{2}y^2 + 2y + \tfrac{3}{2}$$

Now, complete the square!

$$\begin{aligned} x &= -\tfrac{1}{2}(y^2 - 4y) + \tfrac{3}{2} && \triangleleft \text{ factor out leading coeff.} \\ x &= -\tfrac{1}{2}(y^2 - 4y + 4) + 2 + \tfrac{3}{2} && \triangleleft \text{ take } \tfrac{1}{2} \text{ coeff. of } y \text{ and square} \\ x &= -\tfrac{1}{2}(y - 2)^2 + \tfrac{7}{2} \end{aligned}$$

Summary: The equation written in standard form is

$$\boxed{x - \tfrac{7}{2} = -\tfrac{1}{2}(y - 2)^2.}$$

This describes a horizontally oriented parabola that opens left with vertex at $V(\,\tfrac{7}{2}, 2\,)$.

The graph is left to the student, but don't plot too many points. Three are enough!

Solution to (c) Analyze: $2y^2 - 4y + x = 0$.

Write x as a function of y:

$$x = -2y^2 + 4y$$

and complete the square

$$x = -2(y^2 - 2y) \quad \triangleleft \text{ factor out leading coeff.}$$

$$x = -2(y^2 - 2y + 1) + 2 \quad \triangleleft \text{ complete the square}$$

$$x = -2(y - 1)^2 + 2 \quad \triangleleft \text{ write as perfect square}$$

Summary: The parabola written in standard form is

$$\boxed{x - 2 = -2(y - 1)^2.}$$

It opens left with vertex at $V(2, 1)$.

The graph is left to the student.

Ever get the feeling you have done this before?

Solution to (d) Analyze: $2x + 3y + 4y^2 = 1$.

First write the equation as a function:

$$x = -2y^2 - \tfrac{3}{2}y + \tfrac{1}{2}$$

and now, complete the square,

$$\begin{aligned} x &= -2(y^2 + \tfrac{3}{4}y) + \tfrac{1}{2} && \triangleleft \text{ factor out leading coeff.} \\ x &= -2(y^2 + \tfrac{3}{4}y + \tfrac{9}{64}) + \tfrac{9}{32} + \tfrac{1}{2} && \triangleleft \text{ complete square} \\ x &= -2(y + \tfrac{3}{8})^2 + \tfrac{25}{32} \end{aligned}$$

Summary: The equation written in standard form is

$$\boxed{x - \tfrac{25}{32} = -2(y + \tfrac{3}{8})^2.}$$

It opens to the left and has vertex at $V(\,\frac{25}{32}, -\frac{3}{8}\,)$.

The graph is left to the student. Exercise 10.6. ■

10.7. *Solution to* (a) To find where two curves intersect we set the y-coordinates equal and solve for x. In this case, the resulting equation has the second degree, so we can anticipate solving either using the method of factoring, or, most likely, using the Quadratic Formula.

Solution to (a) Find where $y = 2x + 1$ and $y = x^2 + 2x - 1$ intersect. First not that this is the intersection of a line and a parabola. They will either not intersect, intersect at one point, or intersect at two points. (Can you visualize each situation?) Let's see which one it is in this case.

Begin by setting the ordinates equal.

$$x^2 + 2x - 1 = y = 2x + 1 \quad \triangleleft \text{ set ordinates equal}$$

$$x^2 + 2x - 1 = 2x + 1 \quad \triangleleft \text{ eliminate the middle man}$$

$$x^2 = 2 \quad \triangleleft \text{ add } 1 - 2x \text{ to both sides}$$

$$|x| = \sqrt{2} \quad \triangleleft \text{ take square root both sides}$$

$$x = \pm\sqrt{2} \quad \triangleleft \text{ solve absolute equality}$$

Summary: The two curves intersect at two points: When the abscissas are $x = -\sqrt{2}$ and $x = \sqrt{2}$. "Plugging" these values of x back into the straight line, we can calculate the coordinates of intersections:

$$\text{Points of Intersection: } \boxed{(-\sqrt{2}, 1 - 2\sqrt{2}), \quad (\sqrt{2}, 1 + 2\sqrt{2})}$$

Solution to (b) $y = 2x + 1$; $y = x^2 + 4x - 1$

$$x^2 + 4x - 1 = y = 2x + 1 \qquad \triangleleft \text{ equate ordinates}$$

$$x^2 + 4x - 1 = 2x + 1 \qquad \triangleleft \text{ eliminate the middle man}$$

$$x^2 + 2x - 2 = 0 \qquad \triangleleft \text{ sub. } 2x + 1 \text{ from both sides}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-2)}}{2} \qquad \triangleleft \text{ quadratic formula}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= -1 \pm \sqrt{3}$$

Summary: The abscissas of intersection are $\boxed{x = -1 \pm \sqrt{3}}$.

The interested reader can go on to calculate the coordinates of intersection just as I did in part (b).

Solution to (c) $y = x^2 + 2$; $y = 2x^2 - 3x + 3$. This is the intersection of two parabolas that open up.

$$\begin{aligned}
2x^2 - 3x + 3 = y = x^2 + 2 &\quad \triangleleft \text{ equate ordinates} \\
2x^2 - 3x + 3 = x^2 + 2 &\quad \triangleleft \text{ eliminate the middle man} \\
x^2 - 3x + 1 = 0 &\quad \triangleleft \text{ combine similar terms on l.h.s.} \\
x = \frac{3 \pm \sqrt{3^2 - 4}}{2} &\quad \triangleleft \text{ quadratic formula} \\
= \frac{3 \pm \sqrt{5}}{2} &
\end{aligned}$$

Summary: The abscissas of intersection are $\boxed{\dfrac{3 \pm \sqrt{5}}{2}.}$

Solution to (d) $y = x^2 + 2$; $y = 4 - (x-1)^2$. These again are two parabolas, one opening up, the other down.

$$
\begin{aligned}
x^2 + 2 = y &= 4 - (x-1)^2 && \triangleleft \text{ equate ordinates} \\
x^2 + 2 &= 4 - (x-1)^2 && \triangleleft \text{ get rid of } y \\
x^2 + 2 &= 3 + 2x - x^2 && \triangleleft \text{ expand r.h.s.} \\
2x^2 - 2x - 1 &= 0 && \triangleleft \text{ combine similar terms on l.h.s.} \\
x &= \frac{2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)} && \triangleleft \text{ quadratic formula} \\
&= \frac{2 \pm \sqrt{12}}{4} \\
&= \frac{1 \pm \sqrt{3}}{2} && \triangleleft \text{ simplify!}
\end{aligned}
$$

Summary: The abscissas of intersection are $\boxed{x = \frac{1 \pm \sqrt{3}}{2}.}$

Exercise 10.7. ■

10.8. *Solution to* (a) We are given the center C of the circle and one point, P, on the circle. All we need is the radius of the circle. From equation (10) we have

$$d(\,P, C\,) = r$$

or,

$$\boxed{r = d(\,P, C\,).}$$

This boxed formula represents the method of solving these problems.

Solution to (b) We already have the center of the circle, we need the *radius*.

We follow the plan of attack developed in (a):

$$\begin{aligned} r &= d(\,P, C\,) \\ &= \sqrt{(4-1)^2 + (-1-2)^2} \quad \triangleleft \text{ distance formula} \\ &= \sqrt{3^2+3^2} = 3\sqrt{2} \end{aligned}$$

Thus, $r = 3\sqrt{2}$. The equation of the circle is

$$(x-1)^2 + (y-2)^2 = (3\sqrt{2})^2$$

$$\boxed{(x-1)^2 + (y-2)^2 = 18}$$

Solution to (c) As in the part (a), $r = d(P, C)$, where $C(\,-2, 3\,)$ and $P(\,5, 2\,)$.

$$\begin{aligned} r &= d(P, C) \\ &= \sqrt{(5-(-2))^2 + (2-3)^2} \\ &= \sqrt{49+1} = \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

The equation is for the desired circle is

$$(x+2)^2 + (y-3)^2 = (5\sqrt{2})^2$$

or,

$$\boxed{(x+2)^2 + (y-3)^2 = 50}$$

Exercise 10.8. ■

10.9. *Solution*: Since the two points P_1 and P_2 are diametrically opposite, the radius is given by

$$r = \frac{1}{2}\, d(\, P_1, P_2\,)$$

The center of the circle is the midpoint of the line segment P_1P_2; simple use the midpoint formula.

Solution to (b) $P_1(1,3)$ and $P_2(3,5)$.

I follow my own game plan:

$$\begin{aligned} r &= \tfrac{1}{2}d(P_1, P_2) \\ &= \tfrac{1}{2}\sqrt{(1-3)^2+(3-5)^2} \\ &= \tfrac{1}{2}\sqrt{8} = \tfrac{1}{2}(2\sqrt{2}) \\ &= \boxed{\sqrt{2}} \end{aligned}$$

The center is the midpoint between $P_1(1,3)$ and $P_2(3,5)$

$$\begin{aligned} C(h,k) &= C\left(\frac{1+3}{2}, \frac{3+5}{2}\right) \\ &= \boxed{C(2,4)} \end{aligned}$$

The equation of the circle is

$$\boxed{(x-2)^2+(y-4)^2=2}$$

Solution to (c) $P_1(\,-3,1\,)$ and $P_2(\,4,6\,)$

$$\begin{aligned} r &= \tfrac{1}{2}\sqrt{(-3-4)^2+(1-6)^2} \\ &= \tfrac{1}{2}\sqrt{49+25} = \boxed{\tfrac{1}{2}\sqrt{74}} \end{aligned}$$

The center is the midpoint between $P_1(\,-3,1\,)$ and $P_2(\,4,6\,)$:

$$\begin{aligned} C(\,h,k\,) &= C\left(\frac{-3+4}{2},\frac{1+6}{2}\right) \\ &= \boxed{C\left(\,\tfrac{1}{2},\tfrac{7}{2}\,\right)} \end{aligned}$$

The equation of the circle is

$$\boxed{(x-\tfrac{1}{2})^2+(y-\tfrac{7}{2})^2=\tfrac{37}{2}}$$

where we have calculated $r^2 = \frac{1}{4}(74) = \frac{37}{2}$

Exercise 10.9. ■

10.10. *Solution to* (a) $x^2 + y^2 - 6x + 2y - 4 = 0$

We complete the square and interpret the result.

$$\begin{aligned} x^2 + y^2 - 6x + 2y - 4 &= 0 && \triangleleft \text{ given} \\ (x^2 - 6x) + (y^2 + 2y) &= 4 && \triangleleft \text{ group similar terms} \\ (x^2 - 6x + 9) + (y^2 + 2y + 1) &= 4 + 9 + 1 && \triangleleft \text{ complete square} \\ (x - 3)^2 + (y + 1)^2 &= 14 && \triangleleft \text{ center-radius form!} \end{aligned}$$

This last equation is in the center-radius form. It represents a circle with center at $C(3, -1)$ and radius $r = \sqrt{14}$

Solution to (b) $x^2 + y^2 + 4y = 2$

$$\begin{aligned} x^2 + y^2 + 4y &= 2 && \triangleleft \text{ given} \\ x^2 + (y^2 + 4y) &= 2 && \triangleleft \text{ group terms} \\ x^2 + (y^2 + 4y + 4) &= 6 && \triangleleft \text{ complete square} \\ x^2 + (y + 2)^2 &= 6 && \triangleleft \text{ center-radius form!} \end{aligned}$$

This is the equation of a circle with center at $C(0, -2)$ and having radius of $r = \sqrt{6}$.

Solution to (c) $3x^2 + 3y^2 - 8x = 0$

$$
\begin{aligned}
3x^2 + 3y^2 - 8x &= 0 && \triangleleft \text{ given} \\
(3x^2 - 8x) + 3y^2 &= 0 && \triangleleft \text{ group similar terms} \\
3(x^2 - \tfrac{8}{3}x) + 3y^2 &= 0 && \triangleleft \text{ factor out leading coeff.} \\
3(x^2 - \tfrac{8}{3}x + \tfrac{16}{9}) + 3y^2 &= \tfrac{16}{3} && \triangleleft \text{ complete square} \\
3(x - \tfrac{4}{3})^2 + 3y^2 &= \tfrac{16}{3} && \triangleleft \text{ write as perfect square} \\
(x - \tfrac{4}{3})^2 + y^2 &= \tfrac{16}{9} && \triangleleft \text{ center-radius form!}
\end{aligned}
$$

This is the equation of a circle with center at $C(\frac{4}{3}, 0)$ and radius $r = \frac{4}{3}$.

Solution to (d) $x^2 + y^2 + 2y - 4x + 10 = 0$

$$x^2 + y^2 + 2y - 4x + 10 = 0 \quad \triangleleft \text{ given}$$

$$(x^2 - 4x) + (y^2 + 2y) = -10 \quad \triangleleft \text{ group similar terms}$$

$$(x^2 - 4x + 4) + (y^2 + 2y + 1) = -10 + 4 + 1 \quad \triangleleft \text{ complete square}$$

$$(x - 2)^2 + (y + 1)^2 = -5$$

The equation $(x - 2)^2 + (y + 1)^2 = -5$ does *not* describe a circle since the number on the right-hand side is negative. There is no point (x, y) that satisfies this equation since the left-hand side is always nonnegative and the right-hand side is negative. This equation has *no graph*.

Exercise 10.10. ■

10.11. *Solution to* (a) Find the points of intersection between $x^2 + y^2 = 4$ and $y = 2x$. We equate the ordinates and solve for x. The easiest way of doing this is to substitute the value of y into the other equation:

$$x^2 + y^2 = 4 \text{ and } y = 2x$$

$$\begin{aligned} x^2 + (2x)^2 &= 4 && \triangleleft \text{ subst. } y = 2x \text{ into circle} \\ x^2 + 4x^2 &= 4 && \triangleleft \text{ expand} \\ 5x^2 &= 4 && \triangleleft \text{ combine} \\ x &= \pm\frac{2}{\sqrt{5}} && \triangleleft \text{ solve in one step!} \end{aligned}$$

These are the abscissas of intersection. To obtain the corresponding ordinates, simple substitute these values into $y = 2x$ to obtain the coordinates of intersection:

$$\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) \quad \left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$$

Solution to (b) Find the points of intersection between $x^2 + y^2 = 4$ and $y = x + 1$.

$$\begin{aligned} x^2 + (x+1)^2 &= 4 && \triangleleft \text{ substitute } y = x + 1 \text{ into first eq.} \\ x^2 + x^2 + 2x + 1 &= 4 && \triangleleft \text{ expand} \\ 2x^2 + 2x - 3 &= 0 && \triangleleft \text{ combine} \end{aligned}$$

Now apply the QUADRATIC FORMULA:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(2)(-3)}}{4} \\ &= \frac{-2 \pm \sqrt{28}}{4} = \frac{-1 \pm \sqrt{7}}{2} \end{aligned}$$

Summary: The abscissas of intersection are $\boxed{x = \dfrac{-1 \pm \sqrt{7}}{2}.}$

The corresponding y-coordinates (the ordinates) can be obtained by substituting these abscissas into the straight line equation: $y = x + 1$. I leave this to you. ∎

Solution to (c) Find the points of intersection between $x^2+y^2-2x=4$ and $y=2x-1$.

$$\begin{aligned} x^2+(2x-1)^2-2x &= 4 && \triangleleft \text{ substitute in for } y \\ x^2+4x^2-4x+1-2x &= 4 && \triangleleft \text{ expand} \\ 5x^2-6x-3 &= 0 && \triangleleft \text{ combine} \end{aligned}$$

Now apply the QUADRATIC FORMULA:

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36-4(5)(-3)}}{10} \\ &= \frac{6 \pm \sqrt{96}}{10} = \frac{6 \pm 4\sqrt{6}}{10} \\ &= \frac{3 \pm 2\sqrt{6}}{5} \end{aligned}$$

Summary: The abscissas of intersection are $\boxed{x = \frac{3 \pm 2\sqrt{6}}{5}.}$ ■

Solution to (d) Find the points of intersection between $x^2+y^2-2x=4$ and $x+2y=6$.

In this case, I think I'll do something a little different. I'll solve for x in the second equation, $x=6-2y$ and substitute this into the first equation. This is to avoid working with *fractions* when it is really not necessary to do so!

$$
\begin{aligned}
(6-2y)^2+y^2-2(6-2y) &= 4 && \triangleleft \text{ Since } x=6-2y \\
36-24y+4y^2+y^2-12+4y &= 4 && \triangleleft \text{ expand} \\
5y^2-20y+20 &= 0 && \triangleleft \text{ combine} \\
y^2-4y+4 &= 0 && \triangleleft \text{ simplify: Note perfect square!} \\
(y-2)^2 &= 0 && \triangleleft \text{ write is so} \\
y &= 2 && \triangleleft \text{ solve!}
\end{aligned}
$$

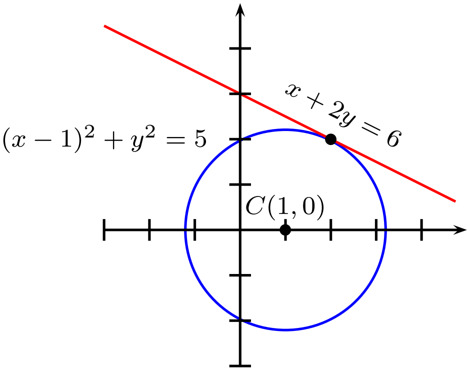
In this case, there is only one point of intersection:

$$y=2 \text{ and } x=6-2y \implies x=2$$

Thus,

$$\boxed{\text{Point of Intersection: } (2, 2)}$$

If you draw the graphs of these two, the circle and the line, you will see that the line is *tangent* to the circle at the point $(2, 2)$.



Never mind, I did it myself!

Exercise 10.11. ■

10.12. *Solutions*: The following answers are given without extensive discussion. If you don't see how I got the number, follow the tip given above.

(a) $\sin(3\pi/4) = \sqrt{2}/2$.

(b) $\cos(7\pi/6) = -\sqrt{3}/2$. (Note: $\dfrac{7\pi}{6} = \pi + \dfrac{\pi}{6}$.)

(c) $\sin(-7\pi/6) = -\sin(7\pi/6) = -(-1/2) = 1/2$. (Note: Same note as previous note.)

(d) $\cos(-21\pi/2) = \cos(21\pi/2) = 0$. (Note: $\dfrac{21\pi}{2} = 10\pi + \dfrac{\pi}{2}$.)

(e) $\sin(9\pi/3) = \sin(3\pi) = 0$.

(f) $\cos(10\pi/3) = -1/2$. (Note: $\dfrac{10\pi}{3} = 3\pi + \dfrac{\pi}{3}$.)

(g) $\sin(10\pi/3) = -\sqrt{3}/2$. (Note: ditto.)

(h) $\cos(-29\pi/6) = \cos(29\pi/6) = -\sqrt{3}/2$. (Note: $\dfrac{29\pi}{6} = 5\pi - \dfrac{\pi}{6}$; i.e., $29\pi/6$ is $\pi/6$ "short" of 5π. Wrap 5π, then come back $\pi/6$.)

Exercise Notes: I could have done part (h) in a way that is consistent with how I did the others:

$$\frac{29\pi}{6} = 4\pi + \frac{5\pi}{6},$$

Therefore,

$$\cos(29\pi/6) = \cos(5\pi/6)$$

Now I would need to compute $\cos(5\pi/6)$. I found it much easier to realize that $29\pi/6$ was $\pi/6$ "short" of 5π.

▪ Though I didn't refer to the property explicitly, what we are in fact doing is using the periodic properties. In the previous remark, when I write

$$\cos(29\pi/6) = \cos(5\pi/6)$$

That is the periodic property of the cosine. To wrap around an amount of $29\pi/6$ you first wrap 4π which is *twice* around the unit circle, then you go another $5\pi/6$.

Exercise 10.12. ■

10.13. The values of x at which $\cos(x)$ takes on its minimum value of $y = -1$ is

$$x = \ldots,\ -5\pi,\ -3\pi,\ -\pi,\ \pi,\ 3\pi,\ 5\pi,\ \ldots$$

In other words, $\cos(x)$ takes on its minimum value of $y = -1$ at all *odd multiples of* π.

Exercise 10.13. ■

10.14. *Solution*: If you studied Table 1 and Figure 9, maybe you formulated the following answer:

$$x = \ldots, -4\pi, \; -3\pi, \; -2\pi, \; \pi, \; 0, \; \pi, 2\pi, \; 3\pi, \; 4\pi, \ldots$$

The function $y = \sin(x)$ is equal to zero at all *integer multiples of* π.

Exercise 10.14. ■

10.15. *Solution*: From TABLE 1 and FIGURE 9 we see that

$$x = \dots, \ -\frac{13\pi}{2}, \ -\frac{9\pi}{2}, \ -\frac{5\pi}{2}, \ -\frac{\pi}{2}, \ \frac{\pi}{2}, \ \frac{5\pi}{2}, \ \frac{9\pi}{2}, \ \frac{13\pi}{2}, \dots$$

Every other odd positive multiple of $\frac{\pi}{2}$, and the negatives of all those numbers.

Did I say a *good english sentence*? Let's try again.

These numbers can be written in a more compact form:

$$x = \frac{\pi}{2} + 2n\pi \quad n \in \mathbb{Z}$$

That is, $\sin(x) = 1$ for all numbers that differ from $\frac{\pi}{2}$ by an integer multiple of 2π.

Is that better?

Do you see the validity of these statement? Exercise 10.15. ■

10.16. *Solution*: From Table 1 and Figure 9 we see that

$$x = \ldots,\ -\frac{15\pi}{2},\ -\frac{11\pi}{2},\ -\frac{7\pi}{2},\ -\frac{3\pi}{2},\ \frac{3\pi}{2},\ \frac{7\pi}{2},\ \frac{11\pi}{2},\ \frac{15\pi}{2},\ldots$$

These numbers can be written in a more compact form:

$$x = \frac{3\pi}{2} + 2n\pi \quad n \in \mathbb{Z}$$

That is, $\sin(x) = -1$ at all numbers that differ from $\dfrac{3\pi}{2}$ by an integer multiple of 2π.

Do you see the validity of the statement? Exercise 10.16. ■

10.17. *Solution*: We have

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Therefore,

$$\begin{aligned} \mathrm{Dom}(\cot) &= \{\, x \mid \sin(x) \neq 0 \,\} \\ &= \{\, x \mid x \neq n\pi, \quad n \in \mathbb{Z} \,\} \end{aligned}$$

◁ from Exercise 10.14

Answer: $\boxed{\mathrm{Dom}(\cot) = \{\, x \mid x \neq n\pi, \quad n \in \mathbb{Z} \,\}}$

Exercise 10.17. ■

10.18. *Answers*:

(a) $\dfrac{2\pi}{3}$ is the same as $\theta = \dfrac{2\pi}{3}\dfrac{180^\circ}{\pi} = 120^\circ$.

(b) $-\dfrac{5\pi}{6}$ is the same as $\theta = -\dfrac{5\pi}{6}\dfrac{180^\circ}{\pi} = -150^\circ$

(c) 3π is the same as $\theta = 3\pi\dfrac{180^\circ}{\pi} = 540^\circ$

(d) $-\dfrac{3\pi}{4}$ is the same as $\theta = -\dfrac{3\pi}{4}\dfrac{180^\circ}{\pi} = -135^\circ$

(e) $\dfrac{5\pi}{2}$ is the same as $\theta = \dfrac{5\pi}{2}\dfrac{180^\circ}{\pi} = 450^\circ$

Exercise 10.18. ■

10.19. *Answers*:

(a) $30°$ is the same as $t = 30° \dfrac{\pi}{180°} = \dfrac{\pi}{6}$.

(b) $60°$ is the same as $t = 60° \dfrac{\pi}{180°} = \dfrac{\pi}{3}$.

(c) $45°$ is the same as $t = 45° \dfrac{\pi}{180°} = \dfrac{\pi}{4}$.

(d) $-300°$ is the same as $t = -300° \dfrac{\pi}{180°} = -\dfrac{5\pi}{3}$.

(e) $225°$ is the same as $t = 225° \dfrac{\pi}{180°} = \dfrac{5\pi}{4}$.

That seemed very easy.

Exercise 10.19. ■

Solutions to Examples

10.1. *Solution*: We begin by completing the square using the rules given in LESSON 7.

$$\begin{aligned} y &= x^2 - 2x + 4 && \triangleleft \text{ given, now associate } x\text{'s} \\ y &= (x^2 - 2x) + 4 && \triangleleft \text{ take } \tfrac{1}{2} \text{ coef. of } x \text{ and square it} \\ y &= (x^2 - 2x + 1) + 4 - 1 && \triangleleft \text{ add and subtract that} \\ y - 3 &= (x - 1)^2 && \triangleleft \text{ square completed!} \end{aligned}$$

The last equation has the form of (6); here, $a = 1$, $h = 1$ and $k = 3$.

This is a parabola that opens up and has vertex at $\boxed{V(1, 3).}$

Rather than accepting the rule for classification, let's use this example to verify the assertion that $V(1, 3)$ is the vertex.

The vertex is the point at which the parabola turns to create a maximum or minimum. Because the coefficient of the square term, $a = 1$, is positive, this vertex should represent a *low point*. We can see this easily from the equation

$$y - 3 = (x - 1)^2$$

or,

$$y = 3 + (x - 1)^2$$

Notice that when $x = 1$, $y = 3$. The vertex is located at an value of $y = 3$ on the vertical axis. Now, how does this compare with the ordinate (the y-coordinate) of any other point on the graph? Observe that

$$y = 3 + (x - 1)^2 \geq 3,$$

because the expression $(x - 1)^2 \geq 0$. But this inequality states, at least in my mind, that the ordinate, y, of a typical point on the graph is 3 or larger. Doesn't this say that the *vertex is the lowest point on on the graph*?

Example 10.1. ■

10.2. *Solution*: We simply complete the square! First though, we put it in the form of equation (6).

$3 = 4x^2 + 3x + 2y$ ◁ Given. Write as $y = ax^2 + bx + c$

$y = \frac{3}{2} - \frac{3}{2}x - 2x^2$ ◁ Done! Now complete square

$y = -2(x^2 + \frac{3}{4}x) + \frac{3}{2}$ ◁ assoc. x's and factor out coeff. of x

$y = -2(x^2 + \frac{3}{4}x + \frac{9}{64}) + \frac{3}{2} + \frac{9}{32}$ ◁ take $\frac{1}{2}$ coeff. of x and square it

$y = -2(x + \frac{3}{8})^2 + \frac{57}{32}$ ◁ done!

It is important that you fully understand each step—some detail was left out. In particular, make sure you understand the second to last step above. Review the examples and exercises that follow the discussion of completion of the square.

Taking the constant in the last equation to the other side of the equation we get

$$y - \frac{57}{32} = -2(x + \frac{3}{8})^2$$

From this we can see that we have a parabola that opens down and whose vertex is located at $\boxed{V(-\frac{3}{8}, \frac{57}{32}).}$

Be sure you understand why the first coordinate is $h = -\frac{3}{8}$ and *not* $\frac{3}{8}$. (Compare with equation (6).) Example 10.2. ■

10.3. *Solution*: Simply apply the technique of completing the square.

Solution to (a) $x^2 + y^2 - 2x + 4y + 1 = 0$.

$$
\begin{aligned}
x^2 + y^2 - 2x + 4y + 1 &= 0 && \triangleleft \text{ given} \\
(x^2 - 2x) + (y^2 + 4y) &= -1 && \triangleleft \text{ group similar terms} \\
(x^2 - 2x + 1) + (y^2 + 4y + 4) &= -1 + 1 + 4 && \triangleleft \text{ complete sq. of each} \\
(x - 1)^2 + (y + 2)^2 &= 4 && \triangleleft \text{ write as perfect squares}
\end{aligned}
$$

Summary: The center-radius form is

$$\boxed{(x - 1)^2 + (y + 2)^2 = 4.}$$

This is the equation of a circle with center at $C(\,1, -2\,)$ and radius of $r = 2$.

Solution to (b) $2x^2 + 2y^2 + 4x - y - 2 = 0$.

$$
\begin{aligned}
2x^2 + 2y^2 + 4x - y - 2 &= 0 && \triangleleft \text{ given} \\
(2x^2 + 4x) + (2y^2 - y) &= 2 && \triangleleft \text{ group similar terms} \\
2(x^2 + 2x) + 2(y^2 - \tfrac{1}{2}y) &= 2 && \triangleleft \text{ factor out leading coeff.} \\
(x^2 + 2x) + (y^2 - \tfrac{1}{2}y) &= 1 && \triangleleft \text{ divide out these coeff.} \\
(x^2 + 2x + 1) + (y^2 - \tfrac{1}{2}y + \tfrac{1}{16}) &= 1 + 1 + \tfrac{1}{16} && \triangleleft \text{ complete square} \\
(x + 1)^2 + (y - \tfrac{1}{4})^2 &= \tfrac{33}{16} && \triangleleft \text{ write as perfect squares}
\end{aligned}
$$

Summary: The equation written in the center-radius form is

$$\boxed{(x + 1)^2 + (y - \tfrac{1}{4})^2 = \tfrac{33}{16}}$$

This is the equation of a circle with center at $C(-1, \frac{1}{4})$ and having radius $r = \frac{\sqrt{33}}{4}$.

Example 10.3. ■

10.4. All the requested values are for $\frac{\pi}{n}$, for $n = 1, 2, 3, 4$, or 6, so we can deduce the requested values using TABLE 1.

(a) Calculate $\sin(-\pi/3)$.

Solution: We can recall the *odd property* of the sine function

$$\sin(-\pi/3) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2} \quad \triangleleft \text{ by TABLE 1}$$

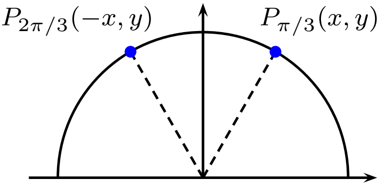
Another way or reasoning is to realize that the ordinate terminal point $P_{-\pi/3}$ is the negative of the ordinate of the terminal point $P_{\pi/3}$ . . . this was the reasoning used to establish the symmetry properties in the first place.

(b) Calculate $\cos(2\pi/3)$.

Solution: This is easy if you realize, from geometric considerations, that the terminal point of $\pi/3$ and $2\pi/3$ are symmetrically placed on opposite of the y-axis. That being the case, the x-coordinates are opposite in sign; therefore,

$$\cos(2\pi/3) = -\cos(\pi/3) = -\frac{1}{2} \quad \triangleleft \text{ by TABLE 1}$$

Depicted below are the points $P_{\pi/3}$ and $P_{2\pi/3}$; notice that $P_{\pi/3}$ is "$\pi/6$" short of the vertical and $P_{2\pi/3}$ is "$\pi/6$" beyond the vertical. This means they are symmetrical with respect to the vertical (the y-axis).



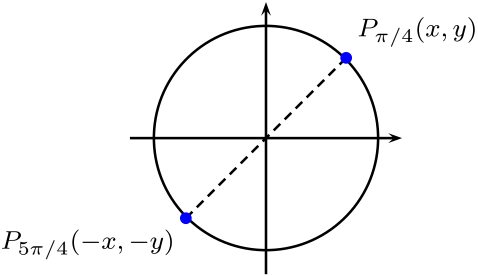
• Question Verify the claim that, "$P_{\pi/3}$ is '$\pi/6$' short of the vertical and $P_{2\pi/3}$ is '$\pi/6$' beyond the vertical."

(c) Calculate $\cos(5\pi/4)$.

Solution: Obviously,

$$\frac{5\pi}{4} = \tfrac{5}{4}\pi = (1\tfrac{1}{4})\pi = (1 + \tfrac{1}{4})\pi = \pi + \frac{\pi}{4}$$

To wrap $t = 5\pi/4$ around the unit circle is equivalent to first wrapping π units, and then wrap $\pi/4$ beyond that. The terminal point, $P_{5\pi/4}$ is in the third quadrant directly opposite the point $P_{\pi/4}$.



As you can see from the picture, both coordinates of $P_{5\pi/4}$ are the *negative* of their counterparts in the point $P_{\pi/4}$; By Table 1 we have $\cos(\pi/4) = \sqrt{2}/2$, we therefore have

$$\cos(5\pi/4) = -\cos(\pi/4) = -\frac{\sqrt{2}}{2}$$

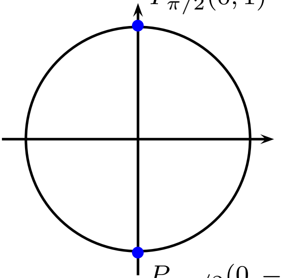
$$\boxed{\cos(5\pi/4) = -\frac{\sqrt{2}}{2}}$$

(d) Calculate $\sin(-\pi/2)$.

Solution: This one is easy,

$$\sin(-\pi/2) = -\sin(\pi/2) = -1 \quad \triangleleft \text{ by TABLE 1}$$

A picture is worth four words,



$$\boxed{\sin(-\pi/2) = -1}$$

Example 10.4. ■

10.5. *Solution*: Recall from equation (20) that we had identified all x at which $\cos(x) = 0$ The $\cos(x) = 0$ at all x such that

$$x = \ldots,\ -\frac{5\pi}{2},\ -\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}, \ldots$$

This can be reformulated into a more compact form

$$x = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$$

That is, $\cos(x) = 0$ all values of x that differ from $\dfrac{\pi}{2}$ by an integer multiple of π.

The domain of $\tan(x)$ is all x, *except* the ones listed above. Thus,

$$\mathrm{Dom}(\tan) = \left\{ x \mid x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \right\}$$

Example 10.5. ■

Important Points

Quiz. Answer each of the following about the parabolas. (Some may require some algebraic simplification.)

1. Which of the following is a parabola defining y as a function of x?

(a) $x + y - y^2 = 2$ (b) $x + y - x^2 = 2$
(c) $xy = 3$ (d) n.o.t.

2. Which of the following is a parabola defining y as a function of x?

(a) $x + y + xy = 0$ (b) $x + y + x^3 = 2$
(c) $(x + y)^2 = 1$ (d) n.o.t.

3. Which of the following is a parabola defining y as a function of x?

(a) $(x + 1)^2 = (x + y)$ (b) $(y + 1)^2 = (x + y)$
(c) $y(x + y) = 1$ (d) n.o.t.

4. Which of the following is a parabola defining y as a function of x?

(a) $2x - 3y = 2$ (b) $(x - y)(x + y) = x$
(c) $(x - y)(x + y) = 1 - y - y^2$ (d) n.o.t.

EndQuiz.

Important Point ■

Quiz. For each of the parabolas below, determine whether is opens up or down. Begin by putting the parabola in the form of a function of x, then use equation (4).

1. $x + y - x^2 = 2$

(a) Opens Up (b) Opens Down

2. $x^2 = x - y$

(a) Opens Up (b) Opens Down

3. $(x + 1)^2 = (x + y)$

(a) Opens Up (b) Opens Down

4. $(x - y)(x + y) = 1 - y - y^2$

(a) Opens Up (b) Opens Down

EndQuiz.

You will recognize these parabolas as the ones that appeared as the right answers in the Quiz in the section on recognition.

Important Point ■

We reproduce the following equations for your convenience.

$$P_0(1,0) \quad P_{\pi/2}(0,1) \quad P_\pi(-1,0) \quad P_{3\pi/2}(0,-1) \quad P_{2\pi}(1,0) \qquad \text{(I-1)}$$

The coordinates of these "easy" points contain the cosines and sines of the corresponding values of t. (Given in the subscript of the point, I might note.)

Quiz. Using equation (I-1), answer each of the following correctly. Passing is 100%.

1. $\cos(0) =$

(a) -1 (b) 0 (c) 1 (d) n.o.t.

2. $\sin(3\pi/2) =$

(a) -1 (b) 0 (c) 1 (d) n.o.t.

3. $\cos(\pi) =$

(a) -1 (b) 0 (c) 1 (d) n.o.t.

4. $\sin(\pi/2) =$

(a) -1 (b) 0 (c) 1 (d) n.o.t.

EndQuiz. Important Point ■

That's right! The answer is **Yes**.

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$

Both of these functions are *undefined* when $\cos(x) = 0$. In both cases

$$\begin{aligned} \text{Dom}(\sec) = \text{Dom}(\tan) &= \{\, x \mid \cos(x) \neq 0\} \\ &= \left\{ x \mid x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \right\} \end{aligned}$$

Important Point ■

Verify the claim that, "(1) $P_{\pi/3}$ is '$\pi/6$' short of the vertical and (2) $P_{2\pi/3}$ is '$\pi/6$' beyond the vertical."

Verification of (1) $\dfrac{\pi}{2} - \dfrac{\pi}{6} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$

Verification of (2) $\dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{4\pi}{6} = \dfrac{2\pi}{3}$. Important Point ■