Some useful facts regarding the Laplace transform.  
For a function \( f(t) \) defined on \([0, \infty)\), the Laplace transform \( \mathcal{L}(f) \) is the function of \( s \) defined through the integral 
\[
\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) \, dt.
\]

Here is an extremely short list of Laplace transforms of common functions:
\[
\begin{align*}
\mathcal{L}(\cos(\omega t))(s) &= \frac{s}{s^2 + \omega^2}, \\
\mathcal{L}(\sin(\omega t))(s) &= \frac{\omega}{s^2 + \omega^2}, \\
\mathcal{L}(\cosh(\omega t))(s) &= \frac{s}{s^2 - \omega^2}, \\
\mathcal{L}(\sinh(\omega t))(s) &= \frac{\omega}{s^2 - \omega^2}, \\
\mathcal{L}(t^n)(s) &= \frac{n!}{s^{n+1}}, \\
\mathcal{L}(e^{at})(s) &= \frac{1}{s-a}.
\end{align*}
\]

The following formulas give the transform of functions related to \( f(t) \) in terms of the transform of \( f(t) \):
\[
\begin{align*}
\mathcal{L}(e^{at}f(t))(s) &= \mathcal{L}(f)(s-a), \\
\mathcal{L}(tf(t))(s) &= -\frac{d}{ds}\mathcal{L}(f)(s), \\
\mathcal{L}(f'(t))(s) &= s\mathcal{L}(f)(s) - f(0), \\
\mathcal{L}(f^{(n)}(t))(s) &= s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0), \\
\mathcal{L}(t^n f(t))(s) &= (-1)^n \frac{d^n}{ds^n}\mathcal{L}(f)(s).
\end{align*}
\]

Example: Find the solution of the following initial value problem:
\[
y''' - y'' - y' + y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 1.
\]

Solution: We apply \( \mathcal{L} \) to the equation above. Writing \( Y(s) = \mathcal{L}(y) \), we obtain
\[
s^3Y - 1 - s^2Y - sY + Y = 0.
\]
This means that
\[
Y(s) = \frac{1}{s^3 - s^2 - s + 1} = \frac{1}{s^2(s-1)-(s-1)} = \frac{1}{(s^2-1)(s-1)} = \frac{1}{(s-1)^2(s+1)}.
\]
To determine \( y(t) \) from here first need to split this into partial fractions. Here is what I get:
\[
\frac{1}{(s-1)^2(s+1)} = \frac{1}{4} \frac{1}{s+1} - \frac{1}{4} \frac{1}{s-1} - \frac{1}{2} \frac{1}{(s-1)^2}.
\]
The inverse transform of the first two terms is easy. For the last we notice that
\[
\frac{1}{(s-1)^2} = -\frac{d}{ds} \frac{1}{s-1} = \mathcal{L}(te^t)(s).
\]
We conclude that
\[ y(t) = \frac{1}{4}e^{-t} - \frac{1}{4}e^t - \frac{1}{2}te^t. \]

Example: Find the anti-transform of
\[ F(s) = \frac{2s + 5}{s^2 + 6s + 34}. \]

Solution: We first write \( F(s) \) in a way amenable to our computations:
\[
\frac{2s + 5}{s^2 + 6s + 34} = \frac{2s + 5}{(s + 3)^2 + 25} = \frac{2(s + 3) - 1}{(s + 3)^2 + 25} \quad \frac{s + 3}{5(s + 3)^2 + 25} - \frac{1}{5} \frac{5}{5(s + 3)^2 + 25}.
\]

From here we obtain
\[
\mathcal{L}^{-1}(F(s)) = 2e^{-3t} \cos(5t) - \frac{1}{5}e^{-3t} \sin(5t).
\]

Convolution.

For two functions \( f, g \) defined on the interval \([0, +\infty)\), the convolution of \( f \) and \( g \), denoted by \( f \ast g \), is the following function:
\[
(f \ast g)(t) = \int_0^t f(x)g(t - x) \, dx.
\]

A key property of the convolution is the following: If \( F(s) = \mathcal{L}(f)(s) \) and \( G(s) = \mathcal{L}(g)(s) \), then
\[
\mathcal{L}(f \ast g)(s) = F(s)G(s),
\]
that is, the Laplace transforms converts a convolution into a product.

Example: Find the anti-transform of
\[ H(s) = \frac{4s}{s^4 + 5s^2 + 4}. \]

Solution: We notice that
\[
H(s) = \frac{4s}{(s^2 + 4)(s^2 + 1)} = 2 \cdot \frac{2}{s^2 + 4} \times \frac{s}{s^2 + 1} = 2\mathcal{L} \left( \sin(2t) \ast \cos(t) \right).
\]

From here we conclude that
\[
h(t) = \mathcal{L}^{-1} \left( \frac{4s}{(s^2 + 4)(s^2 + 1)} \right) = 2 \sin(2t) \ast \cos(t) = 2 \int_0^t \sin(2x) \cos(t - x) \, dx.
\]
Now we need to compute the last integral. In this case the following identities are helpful:

\[
\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta),
\]
and

\[
\sin(2\beta) = 2\sin(\beta) \cos(\beta).
\]

From here we get

\[
\int_0^t \sin(2x) \cos(t - x) \, dx = \int_0^t \sin(2x) \left( \cos(t) \cos(x) + \sin(t) \sin(x) \right) \, dx
\]

\[
= \cos(t) \int_0^t \sin(2x) \cos(x) \, dx + \sin(t) \int_0^t \sin(2x) \sin(x) \, dx
\]

\[
= 2 \cos(t) \int_0^t \cos^2(x) \sin(x) \, dx + 2 \sin(t) \int_0^t \sin^2(x) \cos(x) \, dx.
\]

The last two integrals are easy to compute using a substitution. We obtain

\[
h(t) = \frac{4}{3} \cos(t)(1 - \cos^3(t)) + \frac{4}{3} \sin(t) \sin^3(t).
\]

**Unit Step.**

The unit step function, usually denoted by \( u(t) \), is defined as follows:

\[
u(t) = \begin{cases} 
0 & t < 0 \\
1 & t \geq 0
\end{cases}
\]

Note that, for \( a > 0 \), the function \( u(t - a) \), which is often denoted by \( u_a(t) \), corresponds to

\[
u_a(t) = u(t - a) = \begin{cases} 
0 & t < a \\
1 & t \geq a
\end{cases}
\]

The unit step gives us the following formula, valid for \( a > 0 \):

\[
\mathcal{L}(f(t - a)u(t - a))(s) = e^{-as}\mathcal{L}(f)(s).
\]

In particular, we obtain the following:

\[
\mathcal{L}(u(t - a))(s) = \frac{e^{-as}}{s}.
\]

**Example:** Find the solution of the following initial value problem:

\[
y'' - 6y' + 8y = h(t), \quad y(0) = y'(0) = 0,
\]

where

\[
h(t) = \begin{cases} 
0 & t < 8 \\
(t - 8) & t \geq 8
\end{cases}
\]
Solution: We first need to find a usable expression for the function on the right hand side. In this case the function is \( g(t) = t \), translated by 8 units to the right, that is \( g(t - 8) \), and it is also 0 for \( t < 8 \). In other words,

\[
h(t) = u(t - 8)g(t - 8),
\]

where \( g(t) = t \). hence

\[
\mathcal{L}(h) = \frac{e^{-8s}}{s^2}.
\]

Writing \( Y(s) = \mathcal{L}(y)(s) \), we transform the equation to obtain

\[
(s^2 - 6s + 8)Y = \frac{e^{-8s}}{s^2},
\]

so

\[
Y(s) = \frac{e^{-8s}}{s^2(s^2 - 6s + 8)}.
\]

To obtain the inverse transform of this expression we notice first that

\[
\frac{1}{s^2(s^2 - 6s + 8)} = \frac{1}{s^2(s - 4)(s - 2)}
= \frac{1}{2s^2} \left( \frac{1}{s - 4} - \frac{1}{s - 2} \right)
= \frac{1}{2s^2} \times \frac{1}{s - 4} - \frac{1}{2s^2} \times \frac{1}{s - 2}.
\]

This says that

\[
\frac{1}{s^2(s^2 - 6s + 8)} = \mathcal{L}\left( \frac{1}{2} t * e^{4t} - \frac{1}{2} t * e^{2t} \right).
\]

Here we need to compute the convolution

\[
t * e^{\alpha t} = \int_0^t xe^{\alpha(t-x)}dx = e^{\alpha t} \int_0^t xe^{-\alpha x}dx = -\frac{t}{\alpha} + \frac{1}{\alpha^2}(e^{\alpha t} - 1),
\]

where the I integrated the last integral by parts. So far we have

\[
\frac{1}{s^2(s^2 - 6s + 8)} = \mathcal{L}\left( -\frac{t}{8} + \frac{1}{32}(e^{4t} - 1) + \frac{t}{4} - \frac{1}{8}(e^{2t} - 1) \right).
\]

This finally says that

\[
\frac{e^{-8s}}{s^2(s^2 - 6s + 8)} = \mathcal{L}\left( u(t - 8) \left( -\frac{(t - 8)}{8} + \frac{1}{32}(e^{4(t-8)} - 1) + \frac{(t - 8)}{4} - \frac{1}{8}(e^{2(t-8)} - 1) \right) \right),
\]

which is the same as saying

\[
\mathcal{L}^{-1}\left( \frac{e^{-8s}}{s^2(s^2 - 6s + 8)} \right) = u(t - 8) \left( -\frac{(t - 8)}{8} + \frac{1}{32}(e^{4(t-8)} - 1) + \frac{(t - 8)}{4} - \frac{1}{8}(e^{2(t-8)} - 1) \right).
\]
Derivative of a Transform.

When the relation between two functions is of the form \( g(t) = t^k f(t) \) for some integer \( k \geq 1 \), there is a very simple way to obtain the Laplace transform of \( g \) from that of \( f \):

\[
\mathcal{L} (t^n f(t)) (s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}(f)(s).
\]

**Example:** Find the inverse transform of

\[
F(s) = \ln(s - 2) - \ln(s - 4).
\]

**Solution:** Let us call \( f(t) \) the function with \( \mathcal{L}(f) = F(s) \). Then

\[
\mathcal{L}(tf)(s) = -\frac{d}{ds}F(s) = \frac{1}{s - 2} - \frac{1}{s - 4} = \mathcal{L}(e^{2t} - e^{4t})(s).
\]

This obviously says that

\[
tf(t) = e^{2t} - e^{4t},
\]

or

\[
f(t) = \frac{e^{2t} - e^{4t}}{t}.
\]

Transform of a Periodic function.

A function \( f(t) \) is called periodic of period \( T > 0 \) if \( f(t + T) = f(t) \) for all \( t \). The following formula is valid for \( f(t) \) periodic of period \( T \):

\[
\mathcal{L}(f)(s) = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) \, dt.
\]

**Example:** Compute the Laplace transform of the function in Figure 1, that is, of the function that is periodic, with period \( T = 2 \), and for which

\[
h(t) = \sin \left( \frac{\pi t}{4} \right) \text{ when } 0 \leq t < 2.
\]

**Solution:** We know that \( h(t) \) is periodic with period \( T = 2 \), so we have

\[
\mathcal{L}(h)(s) = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} h(t) \, dt = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} \sin \left( \frac{\pi t}{4} \right) \, dt.
\]

To compute this integral we can integrate by parts twice, or look in a table of integrals:

\[
\int e^{at} \sin(\omega t) \, dt = \frac{e^{at}}{a^2 + \omega^2} (\alpha \sin(\omega t) - \omega \cos(\omega t)).
\]
We will use this formula for $\alpha = -s$ and $\omega = \frac{\pi}{4}$, to obtain
\[
\int_0^2 e^{-st} \sin \left( \frac{\pi t}{4} \right) dt = \frac{e^{-st}}{s^2 + \left( \frac{\pi}{4} \right)^2} \left[ -s \sin \left( \frac{\pi t}{4} \right) - \left( \frac{\pi}{4} \right) \cos \left( \frac{\pi y}{4} \right) \right]_0^2 \]
\[
= \frac{1}{s^2 + \left( \frac{\pi}{4} \right)^2} \left( -se^{-2s} + \frac{\pi}{4} \right) \]
\[
= \frac{4}{(16s^2 + \pi^2)} (-4se^{-2s} + \pi). \]

All this together tells us that
\[
\mathcal{L}(h)(s) = \frac{1}{1 - e^{-2s}} \left( \frac{4}{16s^2 + \pi^2} \right) (-4se^{-2s} + \pi). \]

**Dirac Delta.** The Dirac Delta, or unit impulse function, $\delta(t - t_0)$ for a fixed number $t_0 > 0$, is an object usually associated with the following properties:

(i) $\delta(t - t_0) = 0$ for all $t \neq t_0$

(ii) $\int_{-\infty}^\infty \delta(t - t_0) dt = 1$.

The Laplace transform of $\delta(t - t_0)$ is
\[
\mathcal{L} (\delta(t - t_0)) (s) = e^{-t_0s}. \]

**Example:** Solve the following initial value problem:
\[
y'' + 4y' + 13y = \delta(t - 2) - \delta(t - 4), \quad y(0) = y'(0) = 0. \]

**Solution:** Write $Y(s) = \mathcal{L}(y)(s)$. Applying $\mathcal{L}$ to the equation we obtain
\[
(s^2 + 4s + 13)Y = e^{-2s} - e^{-4s}. \]

This says that
\[
Y(s) = \frac{e^{-2s}}{(s + 2)^2 + 9} - \frac{e^{-4s}}{(s + 2)^2 + 9} = \frac{1}{3} \left( \frac{3e^{-2s}}{(s + 2)^2 + 9} - \frac{3e^{-4s}}{(s + 2)^2 + 9} \right), \]

We next notice that
\[
\mathcal{L} \left( e^{-2t} \sin(3t) \right) (s) = \frac{3}{(s + 2)^2 + 9}. \]

In light of the unit step formula, we obtain
\[
y(t) = \frac{1}{3} \left( e^{-2(t-2)} \sin(3(t - 2)) - e^{-2(t-4)} \sin(3(t - 4)) \right). \]