The method of annihilators uses the notation $D$ for derivatives. So for instance

$$Dy = \frac{dy}{dt}.$$  

Concretely,

$$De^{-3t} = -3e^{-3t}$$

and

$$D\left( e^{2t} \cos(5t) \right) = 2e^{2t} \cos(5t) - 5e^{2t} \sin(5t).$$

We can also consider operations of the form $D - \lambda$ for some constant $\lambda$. This just means

$$(D - \lambda)y = Dy - \lambda y = y' - \lambda y,$$

so for instance

$$(D + 4)e^{7t} = De^{7t} + 4e^{7t} = 7e^{7t} + 4e^{7t} = 11e^{7t}.$$  

Finally we write expressions like $D^2$ and $(D - \lambda)^2$. This means that we apply the operator repeatedly. For instance

$$D^2y = D(Dy) = D(y') = y''.$$ 

The same is valid for the operator $(D - \lambda)^2$. This means

$$(D - \lambda)^2y = (D - \lambda)(D - \lambda)y$$

$$= (D - \lambda)(Dy - \lambda y)$$

$$= (D - \lambda)(y' - \lambda y)$$

$$= D(y' - \lambda y) - \lambda(y' - \lambda y)$$

$$= D(y') - D(\lambda y) - \lambda y' + \lambda^2 y$$

$$= y'' - 2\lambda y' + \lambda^2 y.$$ 

In other words,

$$(D - \lambda)^2y = y'' - 2\lambda y' + \lambda^2 y.$$  

Note that this says that

$$(D - \lambda)^2 = D^2 - 2\lambda D + \lambda^2,$$

that is, $(D - \lambda)^2$ acts just like $(a - b)^2$.

Now we focus on finding particular solutions to non-homogeneous, linear equations of constant coefficients of the form

$$y^{(n)} + \alpha_{n-1}y^{(n-1)} + ... + \alpha_1 y + \alpha_0 y = g(t).$$
We recall first that, to find the solutions of the homogeneous equation

\[ y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_1 y + \alpha_0 y = 0 \]

we must find the roots of the polynomial

\[ \lambda^n + \alpha_{n-1}\lambda^{(n-1)} + \ldots + \alpha_1 \lambda + \alpha_0 = 0. \]

In particular, if \( \lambda \) is a real root of multiplicity \( k \), the solutions we obtain for the equation above are

\[ e^{\lambda t}, \quad te^{\lambda t}, \quad t^2 e^{\lambda t}, \quad \ldots, \quad t^{k-1} e^{\lambda t}. \]

What this really says is that all these functions solve the equation

\[ (D - \lambda)^k y = 0. \]

For example, if \( \lambda = 11 \) and \( k = 2 \), the functions

\[ e^{11t}, \quad \text{and} \quad te^{11t} \]

solve the equation

\[ (D - 11)^2 y = 0. \]

Now we notice that

\[ (D - 11)^2 = D^2 - 22D + 121. \]

This says that the equation

\[ (D - 11)^2 y = 0 \]

corresponds to

\[ y'' - 22y' + 121y = 0 \]

so the functions

\[ e^{11t}, \quad \text{and} \quad te^{11t} \]

both solve the equation

\[ y'' - 22y' + 121y = 0. \]

When the roots are complex, for example if \( \lambda = \rho \pm \omega i \) is a root of multiplicity \( k \) the solutions we get are of the form

\[ e^{\rho t} \cos(\omega t), \quad te^{\rho t} \cos(\omega t), \quad \ldots, \quad t^{k-1} e^{\rho t} \cos(\omega t) \]

and

\[ e^{\rho t} \sin(\omega t), \quad te^{\rho t} \sin(\omega t), \quad \ldots, \quad t^{k-1} e^{\rho t} \sin(\omega t). \]
That $\lambda = \rho \pm \omega i$ is a root of multiplicity $k$ means that the functions above all solve the equation
\[ ((D - \rho)^2 + \omega^2)^k y = 0. \]
So for instance, if $k = 1$ and $\lambda = -3 + 5i$ then the equation is
\[ ((D + 3)^2 + 5^2) y = 0 \]
Since $(D + 3)^2 = D^2 + 6D + 9$, the equation
\[ ((D + 3)^2 + 5^2) y = 0 \]
is also the equation
\[ y'' + 6y' + 34y = 0. \]
In this case the functions
\[ e^{-3t} \sin(5t) \quad \text{and} \quad e^{-3t} \cos(5t) \]
both solve the equation
\[ y'' + 6y' + 34y = 0 \]
and also the equation
\[ ((D + 3)^2 + 5^2) y = 0 \]
because these two equations are the same.

If $k = 2$ and $\lambda = -3 + 5i$, then the equation is longer. In this case we get
\[ ((D + 3)^2 + 5^2)^2 y = 0. \]
Since
\[ ((D + 3)^2 + 5^2)^2 = (D + 3)^4 + 2 \times 5^2 \times (D + 3)^2 + 5^4, \]
we could still expand this into an equation with $y/s$, $y''s$ and so on, but this would be rather long.

Now, when looking for particular solutions of a non-homogeneous equation we sometimes can take advantage of the fact that particular solutions solve some equations. For example, let us consider the equation
\[ y'' - 14y' + 63y = 118e^{7t} \cos(2t) + 177e^{7t} \sin(2t). \]
Here we notice that the annihilator of the right hand side is
\[ (D - 7)^2 + 2^2. \]
This means that our particular solution is

$$y_p = \beta_1 e^{7t} \cos(2t) + \beta_2 e^{7t} \sin(2t),$$

and it also means that the functions $e^{7t} \cos(2t)$ and $e^{7t} \sin(2t)$ both solve the equation

$$(D - 7)^2 + 2^2)y = 0.$$  

We expand this equation to obtain

$$y'' - 14y' + 53y = 0.$$  

So we know that $e^{7t} \cos(2t)$ and $e^{7t} \sin(2t)$ both solve this last equation, and hence that our $y_p$ solves this last equation also. Let us now compare this last equation with the equation we want to solve

$$y'' - 14y' + 63y = 118e^{7t} \cos(2t) + 177e^{7t} \sin(2t).$$

The equation that $y_p$ solves shares the first two terms of the left hand side with the one we want to solve. Hence we do this

$$y'' - 14y' + 53y + 10y = 118e^{7t} \cos(2t) + 177e^{7t} \sin(2t).$$

In other words, we take the left hand side of our equation, which is $y'' - 14y' + 63y$, and write it as a sum of the equation that the $y_p$ solves, which is $y'' - 14y' + 53y$, plus whatever we need to actually get the equation we want to solve. Now the good thing about this is that when we plug in $y_p$ into the left hand side of

$$y'' - 14y' + 53y + 10y = 118e^{7t} \cos(2t) + 177e^{7t} \sin(2t),$$

the first three terms will drop out because

$$y_p'' - 14y_p' + 53y_p = 0.$$  

Hence, if we plug $y_p$ into this equation all we will get is

$$10y_p = 118e^{7t} \cos(2t) + 177e^{7t} \sin(2t),$$

which is really easy to solve.

This does not work always as well as here, but one can always try. For example, if we try to find a particular solution of

$$y'' - 5y' + 6y = te^{4t},$$
we first notice that the anihilator of the right hand side is

\[(D - 4)^2.\]

Since the left hand side has roots \(\lambda = 2, 3\), our particular solution is of the form

\[y_p = \beta_1 e^{4t} + \beta_2 te^{4t}.\]

Note that this also says that \(y_p\) solves the equation

\[y'' - 8y' + 16y = 0.\]

Hence we re-write our equation as

\[y'' - 5y' + 6y = y'' - 8y' + 16y + 3y' - 10y = te^{4t}.\]

Hence when we plug in \(y_p\) here we get

\[3y'_p - 10y_p = te^{4t},\]

which is a bit easier to handle than the original equation.