Question 1 (5 pts). Find two linearly independent solutions of the homogeneous problem

\[ x^2y'' - 7xy' + 41y = 0. \]

Solution: We assume that the solution is of the form \( y(x) = x^m \). Plugging this into the equation leads to the polynomial

\[
m(m - 1) - 7m + 41 = m^2 - 8m + 41 = (m - 4)^2 + 5^2.
\]

This means that \( m = 4 \pm 5i \). Then the solutions are

\[ y_1(x) = x^4 \cos(5 \ln(x)) \quad \text{and} \quad y_1(x) = x^4 \sin(5 \ln(x)). \]

Question 2 (5 pts). Find a particular solution of the non-homogeneous problem

\[ x^2y'' - 4xy' + 6y = x^3. \]

Solution: Again, for the homogeneous problem we set \( y(x) = x^m \). In this case we get

\[
m(m - 1) - 4m + 6 = m^2 - 5m + 6 = (m - 2)(m - 3).
\]

Hence the solutions to the homogeneous problem are \( y_1(x) = x^2 \) and \( y_2(x) = x^3 \). From here we compute the Wronskian:

\[
W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4.
\]

Next we set up the equation to compute the particular solution. This requires us to write it as

\[ y'' + p(x)y' + q(x)y = g(x). \]

In our case we obtain

\[ y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x, \quad \text{so} \quad g(x) = x. \]

From here we compute

\[ u_1' = -\frac{y_2g(x)}{W(x)} = -\frac{x^3x}{x^4} = -1 \quad \text{so} \quad u_1 = -x, \]

and

\[ u_2' = \frac{y_1g(x)}{W(x)} = \frac{x^2x}{x^4} = \frac{1}{x} \quad \text{so} \quad u_2 = \ln(x). \]

Finally we obtain

\[ y_p(x) = u_1y_1 + u_2y_2 = -x^3 + x^3 \ln(x). \]