Question 1 (5 pts). Find the general solution of the differential equation

\[ y'' + 9y' + 20y = 40e^{-9t}. \]

Solution: We first look at the polynomial associated to the left hand side:

\[ \lambda^2 + 9\lambda + 20 = (\lambda + 4)(\lambda + 5) = 0. \]

The roots of this polynomial are \( \lambda = -4 \) and \( \lambda = -5 \). This means that the solution to the homogeneous problem associated to the equation above is \( y_h(t) = \alpha_1 e^{-4t} + \alpha_2 e^{-5t}. \)

Next, we seek a particular solution to the equation above. The annihilator for the right hand side is \( D + 9 \), and the polynomial for this operator, \( \lambda + 9 \), does not have roots in common with the polynomial associated to the left hand side. Hence we seek a particular solution of the form

\[ y_p(t) = Ae^{-9t}. \]

We plug this into our equation to obtain

\[ y_p'' + 9y_p' + 20y_p = 81Ae^{-9t} + 9(-9)Ae^{-9t} + 20Ae^{-9t} = 20Ae^{-9t} = 40e^{-9t}, \]

so \( A = 2 \). This means that the general solution to the equation above is

\[ y(t) = \alpha_1 e^{-4t} + \alpha_2 e^{-5t} + 2e^{-9t}. \]

Question 2 (5 pts). Find the functions one should consider to construct a particular solution of the equation below by the method of annihilators. You do not need to find the particular solution. Just say what functions must be considered to find it.

\[ y'' + 12y' + 40y = t^4 e^{-6t} \cos(2t). \]

Solution: First we notice that the polynomial associated to the left hand side is

\[ \lambda^2 + 12\lambda + 40 = (\lambda + 6)^2 + 4. \]

On the other hand, the annihilator for the right hand side is \( ((D + 6)^2 + 4)^5 \). The polynomial associated to this operator is \( ((\lambda + 6)^2 + 4)^5 \). This has the same roots as the polynomial associated to the left hand side of the equation, so we must look for the particular solution among the solutions of the problem

\[ ((D + 6)^2 + 4)^6 y = 0, \]

that is

\[ te^{-6t} \cos(2t), t^2 e^{-6t} \cos(2t), t^3 e^{-6t} \cos(2t), t^4 e^{-6t} \cos(2t), t^5 e^{-6t} \cos(2t), \]

and

\[ te^{-6t} \sin(2t), t^2 e^{-6t} \sin(2t), t^3 e^{-6t} \sin(2t), t^4 e^{-6t} \sin(2t), t^5 e^{-6t} \sin(2t). \]

Note that we do not need to include the functions \( e^{-6t} \cos(2t) \) or \( e^{-6t} \sin(2t) \) because they solve the homogeneous problem associated to this equation.