Question 1. Use Laplace transform to solve the following initial value problem:
\[ y'' + 5y' - 14y = 0, \quad y(0) = -1, \quad y'(0) = -1. \]

Solution: First we apply the Laplace transform to the equation above. Writing \((\mathcal{L}y)(s) = Y(s)\), we obtain
\[ s^2Y + s + 1 + 5(sY + 1) - 14Y = (s^2 + 5s - 14)Y + s + 6 = 0. \]
This gives us
\[ Y(s) = \frac{-s - 6}{s^2 + 5s - 14}. \]
We next use partial fractions to split this in two fractions that are known Laplace transforms. The result of this is
\[ Y(s) = \frac{-8}{9(s - 2)} + \frac{-1}{9(s + 7)}. \]
This now can be easily inverted. The result is
\[ y(t) = \left(\mathcal{L}^{-1}\right)\left(\frac{-8}{9(s - 2)} + \frac{-1}{9(s + 7)}\right) = \frac{-8}{9}e^{2t} - \frac{1}{9}e^{-7t}. \]

Question 2. Find the inverse Laplace transform of
\[ \frac{2s^2 + 9s + 9}{(s^2 - 18s + 117)(s^2 + 16)} \]

Solution: In this case we use partial fractions as follows:
\[ \frac{2s^2 + 9s + 9}{(s^2 - 18s + 117)(s^2 + 16)} = \frac{As + B}{s^2 - 18s + 117} + \frac{Cs + D}{s^2 + 16}. \]
To find the constants \(A, B, C,\) and \(D,\) we add the fractions above, to obtain
\[ \frac{2s^2 + 9s + 9}{(s^2 - 18s + 117)(s^2 + 16)} = \frac{(As + B)(s^2 + 16) + (Cs + D)(s^2 - 18s + 117)}{(s^2 - 18s + 117)(s^2 + 16)} \]
\[ = \frac{(A + C)s^3 + (B - 18C + D)s^2 + (16A + 117C - 18D)s + (16B + 117D)}{(s^2 - 18s + 117)(s^2 + 16)} \]
This means we need to solve the system
\[ A + C = 0 \]
\[ B - 18C + D = 2 \]
\[ 16A + 117C - 18D = 9 \]
\[ 16B + 117D = 9. \]
Here are the numbers that I get:

\[ A = \frac{-99}{3077}, \quad B = \frac{8919}{3077}, \quad C = \frac{99}{3077} \quad \text{and} \quad D = \frac{-983}{3077}. \]

With this in hand we try to make the expression above look like the Laplace transform of something. To achieve this, we recall that

\[ \mathcal{L}(e^{at}f(t))(s) = (\mathcal{L}f)(s-a). \]

Since

\[ (\mathcal{L} \cos(\omega t))(s) = \frac{s}{s^2 + \omega^2}, \]

we obtain

\[ \mathcal{L}(e^{at}\cos(\omega t))(s) = \frac{s-a}{(s-a)^2 + \omega^2}. \]

Hence we re-arrange things as follows:

\[
\begin{align*}
2s^2 + 9s + 9 & = \frac{As + B}{s^2 - 18s + 117} + \frac{Cs + D}{s^2 + 16} \\
& = \frac{As + B}{(s-9)^2 + 6^2} + \frac{Cs + D}{s^2 + 16} \\
& = \frac{A(s-9) + B + 9A}{(s-9)^2 + 6^2} + \frac{Cs + D}{s^2 + 16} \\
& = \frac{A(s-9)}{(s-9)^2 + 6^2} + \frac{B + 9A}{6} \left( \frac{1}{(s-9)^2 + 6^2} + \frac{2}{s^2 + 16} \right) + \frac{C}{s^2 + 16} + \frac{D}{4} \frac{4}{s^2 + 16}.
\end{align*}
\]

From here we find that

\[
2s^2 + 9s + 9 = \frac{As + B}{s^2 - 18s + 117} + \frac{Cs + D}{s^2 + 16} \\
= \frac{A(s-9)}{(s-9)^2 + 6^2} + \frac{B + 9A}{6} \frac{1}{(s-9)^2 + 6^2} + \frac{C}{s^2 + 16} + \frac{D}{4} \frac{4}{s^2 + 16}.
\]

We know all the constants except for

\[ \frac{B + 9A}{6} = \frac{1338}{3077}. \]

**Problem 3** Find the Laplace transform of

\[ g(t) = 3u_2(t) - 2u_4(t) + 6u_5(t). \]

**Solution:** Let us recall here that

\[ (\mathcal{L}u_c(t)) = (\mathcal{L}u(t-c)) = \frac{e^{-cs}}{s}. \]
From here we compute directly that
\[
(\mathcal{L}g) = 3e^{-2s} - 2e^{-4s} + 6e^{-5s}.
\]

**Problem 4** Find the inverse Laplace transform of
\[
F(s) = \frac{e^{-6s}}{s^2 + 2s - 3}.
\]

**Solution:** We first consider the fraction
\[
\frac{1}{s^2 + 2s - 3} = \frac{1}{(s + 3)(s - 1)}
\]
and split it in partial fractions. This gives us
\[
\frac{1}{s^2 + 2s - 3} = \frac{1}{4} \left( \frac{1}{s - 1} - \frac{1}{s + 3} \right),
\]
so we want to find the inverse transform of
\[
F(s) = \frac{1}{4} \left( \frac{e^{-6s}}{s - 1} - \frac{e^{-6s}}{s + 3} \right).
\]
This gives us
\[
(\mathcal{L}^{-1}F)(t) = \frac{1}{4} \left( u(t - 6)e^{t-6} - u(t - 6)e^{-3(t-6)} \right).
\]

**Problem 5.** Take the Laplace transform of the initial value problem
\[
y'' - 5y' - 6y = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}, \quad y(0) = y'(0) = 0,
\]
and solve for \(Y(s) = (\mathcal{L}y)(s)\). Then find \(y(t)\).

**Solution:** The first thing that we need to do is to obtain a usable expression for the function
\[
h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t \end{cases}
\]
For this we recall that
\[
u(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t \end{cases}
\]
This allows to write
\[
h(t) = u(t) - u(t - 1).
\]
With this we can now compute the Laplace transform of the equation, including the right hand side:
\[
s^2Y - 5sY - 6Y = (s^2 - 5s - 6)Y = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}.
\]
In other words, we obtain

\[ Y(s) = \frac{1 - e^{-s}}{s(s^2 - 5s - 6)}. \]

Next we use partial fractions for

\[ \frac{1}{s(s^2 - 5s - 6)} = \frac{1}{s(s + 1)(s - 6)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s - 6}. \]

This was actually given in the problem. We obtain

\[ \frac{1}{s(s^2 - 5s - 6)} = \frac{1}{s(s + 1)(s - 6)} = \frac{-\frac{1}{6}}{s} + \frac{\frac{1}{7}}{s + 1} + \frac{\frac{1}{42}}{s - 6}. \]

So fan then we have

\[ Y(s) = \frac{-\frac{1}{6}(1 - e^{-s})}{s} + \frac{\frac{1}{7}(1 - e^{-s})}{s + 1} + \frac{\frac{1}{42}(1 - e^{-s})}{s - 6} \]

from where we finally get

\[ y(t) = -\frac{1}{6}(1 - u(t - 1)) + \frac{1}{7} \left( e^{-t} - u(t - 1) e^{-(t-1)} \right) + \frac{1}{42} \left( e^{6t} - u(t - 1) e^{6(t-1)} \right) \]