Problem 1. Find a particular solution to

$$y'' + 6y' + 9y = -\frac{25e^{-3t}}{2(1+t^2)}.$$ 

Solution: First we find the solutions to the homogeneous equation

$$y'' + 6y' + 9y = 0.$$ 

The polynomial associated to this equation is

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2.$$ 

The root is 3 with multiplicity 2. The solutions to the homogeneous problem are

$$y_1 = e^{-3t}$$

and

$$y_2 = te^{-3t}.$$ 

We will need the Wronskian for these two functions. We compute this to obtain

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t}.$$ 

We seek a particular solution of the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$ 

For this to work we need

$$u_1'(t) = -te^{-3t} \times (-25e^{-3t}) = \frac{25t}{2(1+t^2)e^{-6t}} = \frac{25}{4} \frac{2t}{1+t^2}.$$ 

We integrate this to obtain

$$u_1(t) = \frac{25}{4} \ln(1 + t^2).$$ 

We also need

$$u_2' = \frac{e^{-3t} \times 25e^{-3t}}{2(1+t^2)e^{-6t}} = \frac{25 \times 1}{2(1+t^2)}.$$ 

We find that

$$u_2 = \frac{25}{2} \arctan(t).$$
The particular solution we seek is

\[ y_p(t) = \frac{25}{4} \ln(1 + t^2)e^{-3t} + \frac{25}{2} \arctan(t)te^{-3t}. \]

**Problem 2.** Find the solution of the equation

\[ x^2 y'' - 3xy' - 45y = 0 \]

that also satisfies

\[ y(1) = 10 \text{ and } y'(1) = -3. \]

**Solution:** The polynomial associated to this equation is

\[ m(m - 1) - 3m - 45 = m^2 - 4m - 45 = (m - 9)(m + 5). \]

The general solution of this equation is

\[ y(x) = \alpha_1 x^9 + \alpha_2 x^{-5}. \]

Next we determine the value of the constants. We have

\[ y(0) = \alpha_1 + \alpha_2 = 10, \]

and

\[ y'(0) = 9\alpha_1 - 5\alpha_2 = -3. \]

We solve this system to find

\[ 14\alpha_1 = 47, \text{ so } \alpha_1 = \frac{47}{14}, \]

and

\[ 14\alpha_2 = 93 \text{ so } \alpha_2 = \frac{93}{14}. \]

The solution is

\[ y(x) = \frac{47}{14} x^9 + \frac{93}{14} x^{-5}. \]

**Problem 3.** Find \( y \) if

\[ x^2 y'' - 3xy' + 4y = 0 \]

and

\[ y(1) = 6 \text{ and } y'(1) = 2. \]
\textbf{Solution:} We could solve this as usual, but we can also use the change of variables $x = e^t$. We recall that this change of variables transforms the equation

\[
\alpha_2 x^2 \frac{d^2 y}{dx^2} + \alpha_1 x \frac{dy}{dx} + \alpha_0 y = 0
\]

into the equation

\[
\alpha_2 \frac{d^2 y}{dt^2} + (\alpha_1 - \alpha_2) \frac{dy}{dt} + \alpha_0 y = 0.
\]

This means that the equation

\[
x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0
\]

transforms into

\[
\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = 0.
\]

The polynomial associated to this equation is

\[
\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.
\]

The solutions of this equation are $y_1 = e^{2t}$ and $y_2 = te^{2t}$.

Next we notice that the change of variables $x = e^t$ transforms the initial data

\[
y(1) = 6 \quad \text{and} \quad y'(1) = 2,
\]

written in terms of $x$, into the initial conditions

\[
y(0) = 6 \quad \text{and} \quad y'(0) = 2
\]

which are conditions for $t$. The general solution of the equation, in the $t$ variable, is

\[
y(t) = \alpha_1 e^{2t} + \alpha_2 te^{2t}.
\]

We use our initial data to obtain

\[
y(0) = \alpha_1 = 6.
\]

Next,

\[
y'(0) = 2\alpha_1 + \alpha_2 = 2,
\]

so

\[
\alpha_2 = -10.
\]

The solution is

\[
y(t) = 6e^{2t} - 10te^{2t}.
\]
Now we change back to the $x$ variables. For this we use $t = \ln(x)$, which yields

$$y(x) = 6x^2 - 10 \ln(x)x^2.$$

**Problem 4.** Find the solution of

$$x^2 y'' - 4xy' - 14y = x^4$$

and

$$y(1) = 1, \quad y'(1) = 1.$$

**Solution:** First we seek the roots of the polynomial

$$m(m - 1) - 4m - 14 = m^2 - 5m - 14 = (m + 2)(m - 7).$$

The roots are $m = -2$ and $m = 7$. The solutions of the homogeneous problem are

$$y_1 = x^{-2} \quad \text{and} \quad y_2 = x^7.$$

Next, we seek a particular solution. We can use the method of variation of parameters, but in this case we can at least try

$$y_p(x) = \alpha x^4.$$

Plugging this into the equation we obtain

$$x^2 y_p'' - 4xy_p' - 14y_p = (12 - 16 - 14)\alpha x^4 = x^4.$$

This means that $\alpha = -\frac{1}{18}$. This says that the general solution is

$$y(x) = \alpha_1 x^{-2} + \alpha_2 x^7 - \frac{1}{18} x^4.$$

We then use the initial data:

$$y(1) = \alpha_1 + \alpha_2 - \frac{1}{18} = 1$$

and

$$y'(1) = -2\alpha_1 + 7\alpha_2 - \frac{2}{9} = 1.$$

We solve this system to find

$$\alpha_1 = \frac{37}{54} \quad \text{and} \quad \alpha_2 = \frac{10}{27}.$$

The solution is

$$y(x) = \frac{37}{54} x^{-2} + \frac{10}{27} x^7 - \frac{1}{18} x^4.$$
Problem 5. Find the function $y(x)$ if

$$x^2 y'' - 9xy' + 25 = x^6$$

and

$$y(1) = -9, \quad y'(1) = 5.$$ 

Solution: The polynomial is

$$m(m - 1) - 9m + 25 = (m - 5)^2.$$ 

There is only one root, with multiplicity 2. The solutions of the homogeneous problem are

$$y_1(x) = x^5 \quad \text{and} \quad y_2(x) = x^5 \ln(x).$$

Next, we seek a particular solution. First we compute the Wronskian:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} x^5 & x^5 \ln(x) \\ 5x^4 & 5x^4 \ln(x) + x^4 \end{vmatrix} = x^9.$$ 

Next, we seek a particular solution of the form

$$y(x) = u_1 y_1 + u_2 y_2.$$ 

We compute

$$u_2' = \frac{x^5 x^4}{x^9} = 1, \quad \text{so} \quad u_2 = x.$$ 

Also

$$u_1' = \frac{-x^5 \ln(x) x^4}{x^9} = -\ln(x), \quad \text{so} \quad u_1 = -(x \ln(x) - x).$$

The particular solution is

$$y_p = -(x \ln(x) - x) x^5 + x \times x^5 = x^6.$$ 

Finally we impose the initial data. The general solution of our equation is

$$y(x) = \alpha_1 x^5 + \alpha_2 x^5 \ln(x) + x^6.$$ 

From here we find that

$$y(1) = -9 = \alpha_1 + 1, \quad \text{so} \quad \alpha_1 = -10,$$

and

$$y'(1) = 5 = 5\alpha_1 + \alpha_2 + 6, \quad \text{so} \quad \alpha_2 = 49.$$ 

The solution we seek is

$$y(x) = -10x^5 + 49x^5 \ln(x) + x^6.$$
Problem 6. Find a particular solution of the equation
\[ x^2 y'' + 13xy' + 61y = x^{-6}. \]

Solution: The polynomial is
\[ m(m-1) + 13m + 61 = (m+6)^2 + 25. \]
The roots are \( m = -6 \pm 5i \), so the solutions of the homogeneous problem are
\[ y_1(x) = x^{-6} \cos(5 \ln(x)) \quad \text{and} \quad y_2(x) = x^{-6} \sin(5 \ln(x)). \]
We need the Wronskian of these two functions:
\[ W(x) = \begin{vmatrix} x^{-6} \cos(5 \ln(x)) & x^{-6} \sin(5 \ln(x)) \\ -6x^{-7} \cos(5 \ln(x)) - 5x^{-7} \sin(5 \ln(x)) & -6x^{-7} \sin(5 \ln(x)) + 5x^{-7} \cos(5 \ln(x)) \end{vmatrix} = 5x^{-13}. \]
Next, we find
\[ u_2' = \frac{x^{-6} \cos(5 \ln(x)) x^{-8}}{5x^{-13}} = \frac{1}{5} \frac{\cos(5 \ln(x))}{x}, \quad \text{so} \quad u_2(x) = \frac{\sin(5 \ln(x))}{25}, \]
and
\[ u_1' = -\frac{x^{-8} \times x^{-6} \sin(5 \ln(x))}{5x^{-13}} = -\frac{1}{5} \frac{\sin(5 \ln(x))}{x}, \quad \text{so} \quad u_1(x) = \frac{\cos(5 \ln(x))}{25}. \]
The solution we seek is
\[ y_p = u_1 y_1 + u_2 y_2 = \frac{x^{-6}}{25}. \]
Note that we could have also guessed that
\[ y_p = \alpha x^{-6}. \]
Plugging this into the equation leads to the exact same solution.