Problem 1. Find the solution of the equation

\[ 9y'' - 30y' + 9y = 0 \]

with the initial data

\[ y(0) = 1 \quad \text{and} \quad y'(0) = 0. \]

Then, do the same for the initial data

\[ y(0) = 0 \quad \text{and} \quad y'(0) = 1. \]

Finally find the Wronskian of the two solutions you found before.

Solution: We first look at the polynomial

\[ 9\lambda^2 - 30\lambda + 9 = 3(3\lambda^2 - 10\lambda + 3) = 3(3\lambda - 1)(\lambda - 3). \]

The roots are \( \lambda = \frac{1}{3} \) and \( \lambda = 3. \) The general solution is

\[ y(t) = \alpha_1 e^{\frac{t}{3}} + \alpha_2 e^{3t}. \]

We first find \( y_1(t) \) under the conditions

\[ y_1(0) = \alpha_1 + \alpha_2 = 1 \]

and

\[ y'_1(0) = \frac{\alpha_1}{3} + 3\alpha_2 = 0. \]

We solve this system to find \( \alpha_1 = \frac{9}{8} \) and \( \alpha_2 = -\frac{1}{8}. \) The first solution is

\[ y_1(t) = \frac{9}{8} e^{\frac{t}{3}} - \frac{1}{8} e^{3t}. \]

For the second solution we seek \( y_2(t) \) under the conditions

\[ y_2(0) = \alpha_1 + \alpha_2 = 0 \]

and

\[ y'_2(0) = \frac{\alpha_1}{3} + 3\alpha_2 = 1. \]

We solve this system to find

\[ \alpha_1 = -\frac{3}{8} \quad \text{and} \quad \alpha_2 = \frac{3}{8}. \]
The second solution is
\[ y_2(t) = -\frac{3}{8}e^{\frac{t}{3}} + \frac{3}{8}e^{3t}. \]

To compute the Wronskian of these two functions we first recall that
\[ W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix}, \]
or we can also remember that \( W(t) \) satisfies the equation
\[ W'(t) = -p(t)W(t). \]

Here \( p(t) \) is comes from writing the equation in the form
\[ y'' + p(t)y' + q(t)y = 0. \]
In other words, \( p(t) = -\frac{10}{3} \). Furthermore, we have the initial data for \( y_1 \) and \( y_2 \), which gives us
\[ W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \]

This altogether means that \( W(t) \) satisfies
\[ W' = -\frac{10}{3}W \]
along with \( W(0) = 1 \). We can solve this first order linear equation by various methods to find
\[ W(t) = e^{-\frac{10t}{3}}. \]

**Problem 2.** Find the solution of
\[ y'' + 12y' + 85y = 0 \]
that also satisfies \( y(0) = 4 \) and \( y'(0) = 6 \).

**Solution:** The polynomial associated to this equation is
\[ \lambda^2 + 12\lambda + 85 = (\lambda + 6)^2 + 49. \]

Hence the general solution is
\[ y(t) = \alpha_1e^{-6t}\cos(7t) + \alpha_2e^{-6t}\sin(7t). \]

Then we impose
\[ y(0) = \alpha_1 = 4, \]
\[ y'(0) = -6\alpha_1 \cos(7t) + \alpha_2(-6\cos(7t) + 7\sin(7t)) = 6. \]
and

\[ y'(0) = -6y(0) + 7\alpha_2 = 6, \]

so \( \alpha_2 = \frac{30}{7} \). The solution is

\[ y(t) = 4e^{-6t} \cos(7t) + \frac{30}{7}e^{-6t} \sin(7t). \]

**Problem 3.** Find the solution of the equation

\[ 36y'' + 12y' + 50y = 0 \]

that also solves

\[ y(1) = 4 \quad \text{and} \quad y'(1) = 6. \]

**Solution:** The polynomial associated to the equation is

\[ 36\lambda^2 + 12\lambda + 50 = (6\lambda + 1)^2 + 49. \]

The general solution is

\[ y(t) = \alpha_1 e^{-\frac{1}{6}t} \cos(7t) + \alpha_2 e^{-\frac{1}{6}t} \sin(7t). \]

We now impose the initial data as follows:

\[ y(1) = \alpha_1 e^{-\frac{1}{6}} \cos(7) + \alpha_2 e^{-\frac{1}{6}} \sin(7) = 4 \]

and

\[ y'(1) = -\frac{1}{6}y(1) + 7e^{-\frac{1}{6}}(-\alpha_1 \sin(7) + \alpha_2 \cos(7)) = 6. \]

We obtain the system

\[ \alpha_1 \cos(7) + \alpha_2 \sin(7) = 4e^{\frac{1}{6}}, \]

and

\[- \sin(7)\alpha_1 + \cos(7)\alpha_2 = \frac{20}{21}e^{\frac{1}{6}}.\]

We solve this system for instance by multiplying the first equation by \( \cos(7) \), the second by \( \sin(7) \) and subtracting, to obtain

\[ \alpha_1 = e^{\frac{1}{6}}(4 \cos(7) - \frac{20}{21} \sin(7)). \]

Then we multiply the first equation by \( \sin(7) \), the second by \( \cos(7) \) and then add the equations to obtain

\[ \alpha_2 = e^{\frac{1}{6}}(4 \sin(7) + \frac{20}{21} \cos(7)). \]
We obtain the following solution:

\[ y(t) = e^{\frac{1}{6}}(4 \cos(7) - \frac{20}{21} \sin(7))e^{-\frac{t}{6}} \cos(7t) + e^{\frac{1}{6}}(4 \sin(7) + \frac{20}{21} \cos(7))e^{-\frac{t}{6}} \sin(7t). \]

We can re-arrange this solution as follows. First, put the exponentials together like this:

\[ y(t) = (4 \cos(7) - \frac{20}{21} \sin(7))e^{-\frac{(t-1)}{6}} \cos(7t) + (4 \sin(7) + \frac{20}{21} \cos(7))e^{-\frac{(t-1)}{6}} \sin(7t). \]

Then, we re-arrange the sines and cosines as follows:

\[ y(t) = e^{-\frac{(t-1)}{6}} \left(4(\cos(7) \cos(7t) + \sin(7) \sin(7t)) + \frac{20}{21}(\sin(7t) \cos(7) - \cos(7t) \sin(7))\right). \]

Then we recall that

\[ \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta), \]

and

\[ \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta). \]

Because of this we can re-write our solution as

\[ y(t) = e^{-\frac{(t-1)}{6}} \left(4 \cos(7t - 7) + \frac{20}{21} \sin(7t - 7)\right). \]

In other words, our solution is

\[ y(t) = 4e^{-\frac{(t-1)}{6}} \cos(7(t - 1)) + \frac{20}{21}e^{-\frac{(t-1)}{6}} \sin(7(t - 1)). \]

It is interesting to notice that our initial data were

\[ y(1) = 4 \quad \text{and} \quad y'(1) = 6, \]

and it turned out that our solution looks as

\[ y(t) = y^*(t - 1) \]

where

\[ y^*(t) = 4e^{-\frac{t}{6}} \cos(7t) + \frac{20}{21}e^{-\frac{t}{6}} \sin(7t) \]

solves the equation

\[ 36y'' + 12y' + 50y = 0 \]

along with the initial data

\[ y(0) = 4 \quad \text{and} \quad y'(0) = 6. \]

This is a general principle. Whenever we solve the equation

\[ y^{(n)} + \alpha_{n-1}y^{(n-1)} + \ldots + \alpha_1y' + \alpha_0y = 0 \]
along with the initial data

\[ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}, \]

we can first find the solution \( y^* \) of the equation

\[ y^{(n)} + \alpha_{n-1} y^{(n-1)} + \ldots + \alpha_1 y' + \alpha_0 y = 0 \]

along with the initial data

\[ y(0) = y_0, \quad y'(0) = y_1, \quad \ldots, \quad y^{(n-1)}(0) = y_{n-1}, \]

and then take \( y(t) = y^*(t - t_0) \).

**Problem 4.** Find the solution of the equation

\[ y'' + 2y' - 15y = 0 \]

that also has

\[ y(0) = 9 \quad \text{and} \quad y(1) = 2. \]

**Solution:** First we consider the polynomial associated to the equation:

\[ \lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3). \]

The general solution is

\[ y(t) = \alpha_1 e^{-5t} + \alpha_2 e^{3t}. \]

We now need to apply the conditions

\[ y(0) = 9 \quad \text{and} \quad y(1) = 2. \]

Note that these are not initial data, in that they involve two different points, and no derivatives of \( y \), so we cannot apply the principle mentioned in the previous problem. We still impose the conditions at \( t = 0 \) and \( t = 1 \) to obtain

\[ \alpha_1 + \alpha_2 = 9 \]

and

\[ \alpha_1 e^{-5} + \alpha_2 e^{3} = 2. \]

We solve this system to obtain

\[ \alpha_1 = \frac{9e^3 - 2}{e^3 - e^{-5}} = \frac{9e^8 - 2e^5}{e^8 - 1}, \]
and
\[
\alpha_2 = \frac{2e^5 - 9}{e^8 - 1}.
\]

**Problem 5.** Find the solution of the equation
\[
400y'' + 81y = 0
\]
that also satisfies
\[
y(0) = 6 \quad \text{and} \quad y'(0) = -6.
\]
**Solution:** We look at the polynomial for the equation
\[
400\lambda^2 + 81,
\]
which has roots \(\lambda = \pm \frac{9}{20}i\). The general solution is
\[
y(t) = \alpha_1 \cos \left( \frac{9}{20}t \right) + \alpha_2 \sin \left( \frac{9}{20}t \right).
\]
We then use the initial data
\[
y(0) = \alpha_1 = 6,
\]
and
\[
y'(0) = \frac{9}{20} \alpha_2 = -6.
\]
We conclude that
\[
\alpha_1 = 6 \quad \text{and} \quad \alpha_2 = -\frac{120}{9} = -\frac{40}{3}.
\]

**Problem 6.** Find the solution to the equation
\[
y''' - 11y'' + 18y' = 0
\]
along with the initial data
\[
y(0) = 4, y'(0) = 9, y''(0) = 5.
\]
**Solution:** First the polynomial
\[
\lambda^3 - 11\lambda^2 + 18\lambda = \lambda(\lambda - 2)(\lambda - 9).
\]
The roots are 0, 2 and 9. The general solution is
\[
y(x) = \alpha_1 + \alpha_2 e^{2x} + \alpha_3 e^{9x}.
\]
Next we impose the initial data. This gives us the system

\[ \begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\
2\alpha_2 + 9\alpha_3 &= 9 & \text{and} \\
4\alpha_2 + 81\alpha_3 &= 5.
\end{align*} \]

We can first solve the last two equations separately. This subsystem is

\[ \begin{align*}
2\alpha_2 + 9\alpha_3 &= 9 & \text{and} \\
4\alpha_2 + 81\alpha_3 &= 5.
\end{align*} \]

To solve, for instance we multiply the first equation by 9 and subtract the second equation from the first to obtain

\[ 14\alpha_2 = 76, \]

so

\[ \alpha_2 = \frac{38}{7}. \]

Similarly we obtain

\[ \alpha_3 = \frac{-13}{63}. \]

Finally we use these two results in the first equation to conclude

\[ \alpha_1 = \frac{-67}{63}. \]

**Problem 7.** Find the solution of the problem

\[ y''' + 16y' = 0 \]

that satisfies

\[ y(0) = 7, \; y'(0) = 36, \; y''(0) = 64. \]

**Solution:** The polynomial

\[ \lambda^3 + 16\lambda = \lambda(\lambda^2 + 16) \]

has roots 0 and \( \pm 4i \). The general solution is

\[ y(x) = \alpha_1 + \alpha_2 \cos(4x) + \alpha_3 \sin(4x). \]

Next we apply the initial data. We obtain

\[ y(0) = \alpha_1 + \alpha_2 = 7, \]
\[ y'(0) = 4\alpha_3 = 36, \]
and
\[ y''(0) = -16\alpha_2 = 64. \]
We have \( \alpha_3 = 9, \alpha_2 = -4 \) and \( \alpha_1 = 11. \)

**Problem 7.** Find the solution of the problem
\[ y^{(4)} - 4y''' + 4y'' = 0 \]
that satisfies
\[ y(0) = 12, y'(0) = 13, y''(0) = 4, y'''(0) = 0. \]

**Solution:** The polynomial
\[ \lambda^4 - 4\lambda^3 + 4\lambda^2 = \lambda^2(\lambda - 2)^2 \]
has roots 0 and 2, both with multiplicity 2. The general solution is
\[ y(t) = \alpha_1 + \alpha_2 t + \alpha_3 e^{2t} + \alpha_4 te^{2t}. \]
Next we use the initial data:
\[ y(0) = \alpha_1 + \alpha_3 = 12, \]
\[ y'(0) = \alpha_2 + 2\alpha_3 + \alpha_4 = 13, \]
\[ y''(0) = 4\alpha_3 + 4\alpha_4 = 4, \]
and
\[ y'''(0) = 8\alpha_3 + 12\alpha_4 = 0. \]
We simplify a bit to get the system
\[ \alpha_1 + \alpha_3 = 12 \]
\[ \alpha_2 + 2\alpha_3 + \alpha_4 = 13 \]
\[ \alpha_3 + \alpha_4 = 1 \]
\[ 2\alpha_3 + 3\alpha_4 = 0. \]
It is easy to solve the last two equations first. We get
\[ \alpha_3 = 3 \text{ and } \alpha_4 = -2. \]
Then we use the first two equations to get
\[ \alpha_2 = 9 \text{ and } \alpha_1 = 9. \]