Question 1 (10 pts.) Find the solution of the differential equation

\[ y' = 2(1 + y)t \]

that satisfies \( y(0) = 1 \).

Solution: This equation is separable. We integrate

\[
\int \frac{dy}{1 + y} = \int 2t \, dt + C
\]

to obtain

\[
\ln(1 + y) = t^2 + C \quad \text{so} \quad 1 + y(t) = e^{t^2+C} = e^{t^2}e^C.
\]

As before we call \( \lambda = e^C \) so all this gives

\[
y(t) = -1 + \lambda e^{t^2}.
\]

We use the condition \( y(0) = 1 \) to find that \( \lambda = 2 \). The solution we seek is

\[
y(t) = -1 + 2e^{t^2}.
\]

Question 2 (10 pts). Find an explicit solution of the equation

\[
(1 + x^2)y' + 2xy = e^x.
\]

Solution: We can solve this by first writing

\[
y' + \frac{2x}{x^2 + 1}y = \frac{e^x}{1 + x^2}
\]

and then finding

\[
e^{\int p(x) \, dx} = (1 + x^2).
\]

This means that

\[
\frac{d}{dx} \left( (1 + x^2)y(x) \right) = e^x,
\]

so

\[
y(x) = \frac{e^x}{1 + x^2} + \frac{C}{1 + x^2}.
\]

Here \( C \) is a constant of integration.
Question 3 (10 pts). Suppose I go to a place in northern Canada where the temperature is constantly equal to 0 degrees Fahrenheit. Assume the temperature of my body is 97 Fahrenheit when I arrive there. Assume also I loose temperature at a rate proportional to the difference between the temperature of my body and the temperature of the place. If five minutes after my arrival my body is at a temperature of 96 degrees, how long in hours will it take for me to freeze? (that is, for my body to be at 32 degrees Fahrenheit)

Solution: Let \( t \) be time in hours and \( A(t) \) my body temperature. Since the temperature of the place is 0, the difference between the temperature of my body and the temperature of the place is just \( A(t) \). This tells us that the rate of change of \( A(t) \) is proportional to \( A(t) \). In other words \( A(t) \) satisfies the equation

\[
A' = KA,
\]

where \( K \) is a constant we must determine. This equation by the way has the solution

\[
A(t) = A_0 e^{Kt},
\]

where \( A_0 \) is my initial body temperature, that is, \( A_0 = 97 \). To determine \( K \) we use the information that after 5 minutes my temperature is 96 degrees. Now 5 minutes are \( \frac{1}{12} \) of an hour, so

\[
96 = A \left( \frac{1}{12} \right) = 97e^{\frac{K}{12}}, \quad \text{so} \quad K = 12 \ln \left( \frac{96}{97} \right) - 0.124353444426559.
\]

Now that we know \( K \) we determine the time until I freeze solving the following equation for \( t \):

\[
A(t) = 97e^{Kt} = 32, \quad \text{so} \quad t = \frac{\ln \left( \frac{32}{97} \right)}{K} = 8.91792809453382[hrs].
\]

Question 4 (10 pts). Integrate the equation

\[
(1 + e^{t+y}) + (1 + e^{t+y}) \frac{dy}{dt} = 0.
\]

Solution: We compute

\[
\frac{\partial M}{\partial t} = e^{t+y} = \frac{\partial N}{\partial t},
\]

so the equation is exact. Hence we integrate

\[
\psi(t, y) = \int (1 + e^{t+y}) \, dy + g(t) = y + e^{t+y} + g(t).
\]

Now we differentiate this last function by \( t \) and set the result equal to \( M \) to obtain

\[
e^{t+y} + g'(t) = 1 + e^{t+y},
\]

so \( g(t) = t \). The solution can be written implicitly as

\[
\psi(t, y) = t + y + e^{t+y} = C
\]

where \( C \) is an arbitrary constant.
Question 5 (10 pts). The equation below is not exact. Find an integrating factor for it assuming the integrating factor depends only on one variable. You do not need to solve the equation. Just find the integrating factor.

\[(3t + 2y) + t \frac{dy}{dt} = 0.\]

Solution: Let us check first that
\[
\frac{\partial M}{\partial y} = 2 \neq \frac{\partial N}{\partial t} = 1.
\]
This says that the equation is not exact. Let us recall also that we can find an integrating factor that depends only on one variable when either
\[
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \frac{N}{M} \text{ is independent of } y, \quad \text{or} \quad \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} \frac{M}{N} \text{ is independent of } t.
\]
Here we have
\[
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t} \frac{N}{M} = \frac{2 - 1}{t},
\]
which is independent of \(y\). Hence we solve
\[
\frac{1}{\mu} \frac{d\mu}{dt} = \frac{1}{t},
\]
which implies easily that \(\mu(t) = t\).

Question 6 (10 pts). Determine the value of \(\lambda\) for which the vectors \((2, 4, 0), (0, 1, \lambda)\) and \((3, -1, 3)\) are linearly dependent.

Solution: We compute the determinant
\[
\det \begin{pmatrix} 2 & 4 & 0 \\ 0 & 1 & \lambda \\ 3 & -1 & 3 \end{pmatrix} = -\lambda \det \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} = -\lambda(-14) + 3(2) = 14\lambda + 6
\]
and set the result to zero. We find that
\[
14\lambda + 6 = 0
\]
so we find
\[
\lambda = -\frac{6}{14} = -\frac{3}{7}.
\]
The vectors are dependent when \(\lambda = -3/7\).
Question 7 (10 pts). Determine the values of the power $m$ for which the functions $f_1(t) = 1$, $f_2(t) = t$ and $f_3(t) = t^m$ are linearly dependent.

Solution: We compute the Wronskian

$$W(f_1, f_2, f_3)(t) = \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{pmatrix} = \begin{pmatrix} 1 & t & t^m \\ 0 & 1 & mt^{m-1} \\ 0 & 0 & m(m-1)t^{m-2} \end{pmatrix} = m(m-1)t^{m-2}.$$

This determinant is identically zero if either $m = 0$ or $m = 1$.

Question 8 (10 pts). Given that the function $y_1(t) = t$ solves the equation below, find a second solution $y_2(t)$ of this equation that is linearly independent of $y_1(t)$.

$$t^2y'' - ty' + y = 0$$

Solution: We first write the equation as

$$y'' - \frac{1}{t} + \frac{1}{t^2}y = 0.$$

We identify from here $p(t) = -\frac{1}{t}$ and then

$$e^{-\int p(t) \, dt} = t.$$

We find the second solution $y_2$ by the formula

$$\frac{d}{dt} \left( \frac{y_2}{y_1} \right) = e^{-\int p(t) \, dt} \frac{y_2}{y_1^2} = \frac{1}{t}.$$

Integrating this last equation we obtain

$$\frac{y_2}{y_1} = \ln(t),$$

so we find $y_2(t) = t \ln(t)$. 